

On the ground states and dynamics of space fractional nonlinear Schrödinger/Gross-Pitaevskii equations with rotation term and nonlocal nonlinear interactions

Xavier ANTOINE^a, Qinglin TANG^{a,b,*}, Yong ZHANG^{c,d}

^a*Institut Elie Cartan de Lorraine, Université de Lorraine, Inria Nancy-Grand Est, F-54506 Vandoeuvre-lès-Nancy Cedex, France.*

^b*Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, Technopôle du Madrillet, 76801 Saint-Etienne-du-Rouvray, France.*

^c*Université de Rennes 1, IRMAR, Campus de Beaulieu, 35042 Rennes Cedex, France*

^d*Wolfgang Pauli Institute c/o Fak. Mathematik, University Wien, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria*

Abstract

In this paper, we propose some efficient and robust numerical methods to compute the ground states and dynamics of Fractional Schrödinger Equation (FSE) with a rotation term and nonlocal nonlinear interactions. In particular, a newly developed Gaussian-sum (GauSum) solver is used for the nonlocal interaction evaluation [33]. To compute the ground states, we integrate the preconditioned Krylov subspace pseudo-spectral method [5] and the GauSum solver. For the dynamics simulation, using the rotating Lagrangian coordinates transform [16], we first reformulate the FSE into a new equation without rotation. Then, a time-splitting pseudo-spectral scheme incorporated with the GauSum solver is proposed to simulate the new FSE. In parallel to the numerical schemes, we also prove some existence and nonexistence results for the ground states. Dynamical laws of some standard quantities, including the mass, energy, angular momentum and the center of mass, are stated. The ground states properties with respect to the fractional order and/or rotating frequencies, dynamics involving decoherence and turbulence together with some interesting phenomena are reported.

Keywords: fractional Schrödinger equation, rotation, nonlocal nonlinear interaction, rotating Lagrangian coordinates, Gaussian-sum solver, ground state, dynamics

Contents

1	Introduction	2
2	Brief review of the Gaussian-Sum (GauSum) method	5
3	Ground state computation: properties, numerical scheme and simulations	7
3.1	Existence and nonexistence of the ground states	7
3.2	Numerical method	10
3.3	Numerical results	11

*Corresponding author.

Email addresses: xavier.antoine@univ-lorraine.fr (Xavier ANTOINE), qinglin.tang@inria.fr (Qinglin TANG), yong.zhang@univ-rennes1.fr (Yong ZHANG)

URL: <http://iecl.univ-lorraine.fr/~xantoine/> (Xavier ANTOINE)

4	Dynamics computation: properties, numerical scheme and simulations	13
4.1	Dynamical properties	14
4.2	Numerical method	16
4.2.1	Rotating Lagrangian coordinates transformation	16
4.2.2	A time-splitting pseudo-spectral method	16
4.3	Numerical results	18
5	Conclusion	20
Appendix A	Proof of lemma 4.1	22
Appendix B	Proof of lemma 4.2	23

1. Introduction

Recently, a great deal of attention has been directed towards the derivation of a powerful generalization of PDEs through the inclusion of fractional order operators. These developments now impact strongly most areas of physics and engineering [41, 44, 49, 64, 79]. Additionally, some new applications are also emerging in biology, molecular dynamics, finance, etc. Due to the fact that extremely important applications are related to these models, a significative effort has been made in the last few years to obtain some mathematical properties and numerical tools [41] for the generalized systems of PDEs. An example of such a keen interest is the recent Journal of Computational Physics [49] special issue in 2015 that is dedicated to “Fractional PDEs: Theory, Numerics, and Applications”. The aim of this paper is to contribute to this new hot area for fractional quantum physics, with possible applications, e.g. in Bose-Einstein condensation (BEC).

During the last decades, the classical Schrödinger Equation (SE) has been widely investigated and applied to many areas in physics (optics, electromagnetic, superfluidity, etc.). It is known as the fundamental equation of classical quantum mechanics which can be interpreted by the Feynman path integral approach over Brownian-like quantum paths [35]. Brownian motion (Wiener process) represents a simple diffusion random walk process. More general and complex stochastic processes (Lévy motion) exist and can be still presumably modeled by modifying the standard diffusion equation using a fractional Laplacian operator $(-\Delta)^s := (-\nabla^2)^s$ (with $s > 0$ being the fractional order) [60, 70]. Lévy processes provide a general framework to study *anomalous diffusion*. Fractional diffusion has been widely studied by many authors [22, 29, 44, 60, 61, 64, 74, 79] and is now considered as a suitable way to describe spatially disordered systems (such as porous media and fractal media), turbulent fluids and plasmas, biological media with traps, etc. Fractional diffusion for classical mechanics is referred to as *subdiffusion* if $s < 1$ and as *superdiffusion* if $s > 1$. For more details, we refer to [44] where a concise table shows the scaling laws for fractional diffusion.

Analogously, in the context of quantum mechanics, fractional quantum models, based in particular on Schrödinger-type equations, are now emerging while being however more limited in terms of publications and studies compared with classical fractional mechanics. Laskin extended the Feynman path integral approach over Lévy-like quantum paths and derived a Fractional Schrödinger Equation (FSE) that modifies the SE by involving the fractional Laplacian $(-\Delta)^s$ [53, 54, 55, 56]. The FSE was applied to represent the Bohr atom, fractional oscillator [56], and it is a new fractional approach to study the quantum chromodynamics (QCD) problem of quarkonium [53]. The FSE also arises in the continuum limit of the discrete SE with long-range dispersive interaction [50], in the mathematical description of boson stars [30] and in some models of water wave dynamics [46]. It has also been proposed to study BEC of which the particles obey a non-Gaussian distribution law [32, 72, 73], where FSE was named as Fractional Gross-Pitaevskii Equation (FGPE) and BEC as Fractional BEC (FBEC). Compared with the SE, the literature on FSE is quite limited but growing quickly to understand its mathematical and physical properties.

More precisely, we consider here the following generalized dimensionless (space-)Fractional NonLinear

Schrödinger equation (FNLSE) with a rotation term and a nonlocal nonlinear interaction

$$i\partial_t\psi(\mathbf{x},t) = \left[\frac{1}{2}(-\nabla^2 + m^2)^s + V(\mathbf{x}) + \beta|\psi(\mathbf{x},t)|^2 + \lambda\Phi(\mathbf{x},t) - \Omega L_z \right] \psi(\mathbf{x},t), \quad (1.1)$$

$$\Phi(\mathbf{x},t) = \mathcal{U} * |\psi(\mathbf{x},t)|^2, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \quad d \geq 2. \quad (1.2)$$

In the context of BEC, this equation is also called as FGPE. Here, $\psi(\mathbf{x},t)$ is the complex-valued wavefunction, $t > 0$ is the time variable and $\mathbf{x} \in \mathbb{R}^d$ is the spatial coordinate. The constant $m \geq 0$ denotes the scaled particle mass, with $m = 0$ representing the massless particle. The parameter $s > 0$ is the space fractional order characterizing the nonlocal dispersive interaction. The fractional kinetic operator is defined *via* a Fourier integral operator

$$(-\nabla^2 + m^2)^s \psi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\psi}(\mathbf{k}) (|\mathbf{k}|^2 + m^2)^s e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}, \quad (1.3)$$

where the Fourier transform is given by $\widehat{\psi}(\mathbf{k}) = \int_{\mathbb{R}^d} \psi(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}$. The potential $V(\mathbf{x})$ is supposed to be trapping, a standard example is the harmonic potential given by

$$V(\mathbf{x}) = \begin{cases} \frac{\gamma_x^2 x^2 + \gamma_y^2 y^2}{2}, & d = 2, \\ \frac{\gamma_x^2 x^2 + \gamma_y^2 y^2 + \gamma_z^2 z^2}{2}, & d = 3, \end{cases} \quad (1.4)$$

where γ_v ($v = x, y, z$) is the trapping frequency in the v -direction. The real-valued constants β and λ characterize the local and nonlocal interaction strengths (positive/negative for repulsive/attractive interaction), respectively. The local interaction is supposed to be cubic, but other choices may also be considered. Concerning the nonlocal interaction (1.2), the convolution kernel $\mathcal{U}(\mathbf{x})$ can be chosen as either the kernel of a Coulomb-type interaction or a Dipole-Dipole Interaction (DDI) [15, 17, 24]

$$\mathcal{U}(\mathbf{x}) = \begin{cases} \frac{1}{2^{d-1}\pi|\mathbf{x}|^\mu}, & 0 < \mu \leq d-1, \\ -\delta(\mathbf{x}) - 3\partial_{\mathbf{n}\mathbf{n}}\left(\frac{1}{4\pi|\mathbf{x}|}\right), & \\ -\frac{3}{2}(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\nabla_\perp^2)\left(\frac{1}{2\pi|\mathbf{x}|}\right), & \end{cases} \iff \widehat{\mathcal{U}}(\mathbf{k}) = \begin{cases} \frac{C}{|\mathbf{k}|^{d-\mu}}, & 0 < \mu \leq d-1, \quad \text{Coulomb}, \\ -1 + \frac{3(\mathbf{n} \cdot \mathbf{k})^2}{|\mathbf{k}|^2}, & \text{3D DDI}, \\ \frac{3[(\mathbf{n}_\perp \cdot \mathbf{k})^2 - n_3^2|\mathbf{k}|^2]}{2|\mathbf{k}|}, & \text{2D DDI}, \end{cases} \quad (1.5)$$

where $C = \pi^{d/2-1}2^{1-\mu}\Gamma(\frac{d-\mu}{2})/\Gamma(\frac{\mu}{2})$ ($\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}dx$ is the Gamma function), $\mathbf{n} = (n_1, n_2, n_3)^T \in \mathbb{R}^3$ is a unit vector representing the dipole orientation and $\mathbf{n}_\perp = (n_1, n_2)^T$. In addition, $L_z = -i(x\partial_y - y\partial_x) = -i\partial_\theta$ is the z -component of the angular momentum, Ω represents the rotating frequency.

The FNLSE conserves two important physical quantities (see Section 4.1): the *mass*

$$\mathcal{N}(\psi(\cdot, t)) := \mathcal{N}(t) := \int_{\mathbb{R}^d} |\psi(\mathbf{x}, t)|^2 d\mathbf{x} \equiv \mathcal{N}(0), \quad (1.6)$$

and the *energy*

$$\mathcal{E}(\psi(\cdot, t)) := \mathcal{E}(t) = \int_{\mathbb{R}^d} \left[\frac{1}{2}\bar{\psi}(-\nabla^2 + m^2)^s \psi + V(\mathbf{x})|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\lambda}{2}\Phi|\psi|^2 - \Omega\bar{\psi}L_z\psi \right] \equiv \mathcal{E}(0). \quad (1.7)$$

Here, $\bar{\psi}$ is the complex conjugate of ψ . The ground states $\phi_g(\mathbf{x})$ of the FNLSE (1.1) are defined by

$$\phi_g(\mathbf{x}) = \arg \min_{\phi \in S} \mathcal{E}(\phi), \quad S = \{\phi \in \mathbb{C} \mid \|\phi\|_2 = 1, \mathcal{E}(\phi) < \infty\}, \quad (1.8)$$

where $\|\phi\|_2$ is the $L^2(\mathbb{R}^d)$ -norm of ϕ .

The FNLSE (1.1) brings together a wide range of Schrödinger-type PDEs. When $s = 1$ and $m = 0$, FNLSE reduces to the standard nonlinear Schrödinger equation (NLSE, also known as GPE). Both the

ground states and dynamics properties of NLSE have been extensively studied theoretically and numerically. One can refer e.g. to [3, 4, 6, 7, 8, 11, 14, 15, 16, 17, 20, 23, 28] and references therein. For $s = 1/2$ and Φ taken as the Coulomb potential, (1.1) reduces to the semi-relativistic Hartree equation that models boson stars [30]. Properties of the ground states have been partially investigated in [27, 58, 59] for $V(\mathbf{x}) \equiv 0$. The Cauchy problem of generalized semi-relativistic Hartree equation (with $s \in [\frac{1}{2}, 1]$) has also been widely studied in [1, 24, 26, 30, 37, 57]. To the best of our knowledge, there are neither theoretical nor numerical studies on the ground state properties for $s > 0$ other than the cases $s = 1/2$ and 1. When $s \in (0, 1)$ (which would correspond to a *subdispersion* effect in analogy to the subdiffusion process characterizing heat-like equations [44]) and $m = \lambda = \Omega = 0$, it reduces to the FNLSE that is originally derived by Laskin [53]. Later, he proved the hermitian character of the fractional Hamiltonian, derived the energy spectra of a hydrogen-like atom and computed a fractional oscillator [53, 55, 56]. Since then, the FNLSE has attracted an increasing attention. For example, for stationary FNLSE with bounded/unbounded potential and various generalised nonlinearities other than $|\psi|^2\psi$, the existence of solutions (such as the bound/ground state solutions and radially symmetric solutions), and their corresponding properties have been investigated. Moreover, the global and/or local well-posedness for the initial value problem (1.1)-(1.2) with $V(\mathbf{x}) \equiv 0$ and $\lambda = 0$ were also studied. We refer to [25, 34, 40, 45, 65, 66, 67, 68] and references therein for more details. For $s > 1$ (that we call *superdispersion* hereafter), there are a few Schrödinger-type equations, while it is quite common for the superdiffusion equations [44]. We consider here this case for some possible eventual physical applications.

Generally speaking, it is difficult to obtain analytical solutions of the FNLSE due to the nonlocal fractional dispersive interaction. For example, even for the simplest case with a box potential, there is still a controversy over the eigenpair solutions [18, 19, 42, 43, 47]. Therefore, being able to develop some accurate numerical methods is crucial and would provide a powerful tool to understand fractional quantum mechanics in view of applications. Nevertheless, there are few numerical studies so far. Amore *et al.* [2] proposed a collocation method and Wang *et al.* [75] developed an energy conserving Crank-Nicolson finite difference (FD) scheme when $\Omega = \lambda = 0$. Similar FD schemes were also proposed for coupled equations [76, 77]. As is well-known, the Crank-Nicolson scheme is nonlinearly implicit and hence requires heavy inner iterations. Worse still, the nonlocal nature of the fractional Laplacian naturally leads to dense matrix representation that hinders efficient computations. Recently, the time-splitting Fourier pseudo-spectral method was adapted to study the dynamics when $\Omega = \lambda = 0$ [51, 52]. Decoherence properties and finite time blow-up results were studied respectively in [51] and [52]. When the nonlocal nonlinear interaction ($\lambda \neq 0$) is taken into account, Bao and Dong [13] proposed a sine pseudo-spectral method to compute the ground states and dynamics of the three-dimensional semi-relativistic Hartree equation ($\mu = 1$ in (1.5)). In [13], the Coulomb potential Φ (1.2) is reformulated to satisfy the following Poisson equation

$$-\Delta\Phi = |\psi|^2, \quad \mathbf{x} \in \mathbb{R}^3, \quad \text{with} \quad \lim_{|\mathbf{x}| \rightarrow \infty} \Phi(\mathbf{x}) = 0. \quad (1.9)$$

Similar ideas were also applied to nonlocal DDI in NLSE [11, 12]. However, due to the slow decay property of Φ at the far-field, a quite large computational domain is necessary to guarantee a satisfactory accuracy. Up to now, most existing numerical methods are proposed for non-rotating FNLSE with $s \leq 1$. As far as we know, there were neither theoretical nor numerical methods for the generalized FNLSE (1.1) for both subdispersion $s < 1$ and superdispersion $s > 1$, with $\lambda\Omega \neq 0$. The difficulties to develop an accurate and efficient scheme lie in the evaluation of the nonlocal interaction Φ (1.2) and proper treatment of the rotation term $L_z\psi$.

To compute the nonlocal interaction, Jiang *et al.* [48] recently proposed an accurate NonUniform Fast Fourier Transform (NUFFT)-based algorithm in the Fourier domain by adopting the polar/spherical coordinates near the singularity. The method requires $O(N \log N)$ arithmetic operations (N being the total number of grid points) and is more accurate than the standard PDE approach (1.9). This solver has been recently integrated within the gradient flow algorithm and time-splitting scheme for computing the ground state and dynamics of NLSE [15, 17]. However, this solver is not ideal because of the large pre-factor in $O(N \log N)$, and it is rather slow for 3D problems. Very recently, by approximating the kernel $\mathcal{U}(\mathbf{x})$ with the summation of a finite number of Gaussians, Zhang *et al.* [33] proposed a Gaussian-sum (GauSum)-

based method to evaluate Φ in the physical space. The algorithm also achieves a spectral accuracy, requires $O(N \log N)$ operations and obtains a speed-up factor around 3-5 compared with the NUFFT-based algorithm. Concerning the rotation term, Antoine and Duboscq [5, 8] proposed a robust preconditioned Krylov subspace spectral solver for the ground state computation of the NLSE with large Ω and β . For the dynamics of the NLSE with a rotation term, Bao *et al.* [16] developed a rotating Lagrangian coordinates transformation method to reformulate the rotating term into a time-dependent trapping potential in the rotating Lagrangian coordinates, which allows for the implementation of high-order time-splitting schemes for the new NLSE [21].

The main objectives of this paper are threefold.

1. Investigate theoretically the existence of the ground states of the general FNLSE (1.1) with respect to the fractional order s and the rotation speed Ω . Develop the dynamical laws for the centre of mass as well as other standard dynamical quantities for general s and arbitrary Ω , and compare them with the ones derived in [51].
2. Develop some efficient and accurate numerical methods for computing the ground states and dynamics of the general FNLSE (1.1) by incorporating the GauSum solver into the adapted version of the gradient flow and time-splitting Fourier pseudo-spectral method. The preconditioned Krylov subspace iteration [5] and the rotating Lagrangian transformation technique [16] will be also integrated into the numerical methods for the ground state computation and dynamics simulation, respectively.
3. Apply our numerical methods to study some interesting behavior, such as the influence of the nonlocal dispersion on the ground states and the vortex pattern as well as possible dynamical properties such as chaos and decoherence.

The rest of the paper is organized as follows. In Section 2, we briefly review the Gaussian Sum method. The ground state computation, including the ground states properties and numerical methods as well as numerical results are presented in Section 3. In Section 4, we derive some dynamical laws for some global physical quantities that are usually considered for the standard NLSE. We then propose an efficient and robust numerical method for the dynamics simulation. Some numerical results are also reported. Finally, a conclusion and some discussions are developed in Section 5.

2. Brief review of the Gaussian-Sum (GauSum) method

With the strong confining potential, the density is smooth and decays exponentially fast. Therefore, we can reasonably truncate the whole space to a bounded domain, e.g., a square box $\mathbf{B}_L := [-L, L]^d$. The density is then rescaled to be compactly supported in a unit box \mathbf{B}_1 , which is now the computational domain. One of the key ideas is to use a GauSum approximation of the kernel \mathcal{U} (see U_{GS} in (2.5)) to reformulate the potential into two integrals, namely, the *long-range regular integral* and the *short-range singular integral*. To be precise, we can reformulate the potential (1.2) as follows

$$\Phi(\mathbf{x}) \approx \int_{\mathbf{B}_1} \mathcal{U}(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) d\mathbf{y} = \int_{\mathbf{B}_2} \mathcal{U}(\mathbf{y}) \rho(\mathbf{x} - \mathbf{y}) d\mathbf{y} \quad (2.1)$$

$$= \int_{\mathbf{B}_2} U_{GS}(\mathbf{y}) \rho(\mathbf{x} - \mathbf{y}) d\mathbf{y} + \int_{\mathcal{B}_\delta} (\mathcal{U}(\mathbf{y}) - U_{GS}(\mathbf{y})) \rho(\mathbf{x} - \mathbf{y}) d\mathbf{y} + I_\delta \quad (2.2)$$

$$:= I_1(\mathbf{x}) + I_2(\mathbf{x}) + I_\delta, \quad \mathbf{x} \in \mathbf{B}_1, \quad (2.3)$$

where

$$I_\delta = \int_{\mathbf{B}_2 \setminus \mathcal{B}_\delta} (\mathcal{U}(\mathbf{y}) - U_{GS}(\mathbf{y})) \rho(\mathbf{x} - \mathbf{y}) d\mathbf{y}, \quad (2.4)$$

with $\mathcal{B}_\delta := \{\mathbf{x} \mid |\mathbf{x}| \leq \delta\}$ being a small neighbourhood of the origin with radius $\delta \sim 10^{-4} - 10^{-3}$ and U_{GS} is given explicitly as follows

$$U_{\text{GS}}(\mathbf{y}) = U_{\text{GS}}(|\mathbf{y}|) := \sum_{q=0}^Q w_q e^{-\tau_q^2 |\mathbf{y}|^2}, \quad Q \in \mathbb{N}^+, \quad (2.5)$$

with weights and nodes $\{(w_q, \tau_q)\}_{q=0}^Q$. Here, U_{GS} designates an accurate approximation of \mathcal{U} , up to $\varepsilon_0 \sim 10^{-14} - 10^{-16}$, within the interval $[\delta, 2]$, i.e.

$$\|\mathcal{U}(r) - U_{\text{GS}}(r)\|_{L^\infty([\delta, 2])} \leq \varepsilon_0. \quad (2.6)$$

For (2.4), we have $|I_\delta| \leq C\varepsilon_0 \delta^d \|\rho\|_{L^\infty}$. Thus the remainder integral I_δ is negligible and is omitted here. Note that the GauSum approximation can be numerically computed with *sinc quadrature* and we refer to [33] for more details.

To compute the *regular integral* I_1 , plugging the explicit GauSum approximation (2.5) into $I_1(\mathbf{x})$ yields

$$I_1(\mathbf{x}) = \sum_{q=0}^Q w_q \int_{\mathbf{B}_2} e^{-\tau_q^2 |\mathbf{y}|^2} \rho(\mathbf{x} - \mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbf{B}_1. \quad (2.7)$$

For $\mathbf{x} \in \mathbf{B}_1$ and $\mathbf{y} \in \mathbf{B}_2$, we have $\mathbf{x} - \mathbf{y} \in \mathbf{B}_3$ and we can approximate the density on \mathbf{B}_3 by finite Fourier series. More specifically, the density ρ is well approximated by Fourier series after zero-padding to \mathbf{B}_3 as follows

$$\rho(\mathbf{z}) \approx \sum_{\mathbf{k}} \hat{\rho}_{\mathbf{k}} \prod_{j=1}^d e^{\frac{2\pi i k_j}{b_j - a_j} (z^{(j)} - a_j)}, \quad \mathbf{z} = (z^{(1)}, \dots, z^{(d)}) \in \mathbf{B}_3, \quad (2.8)$$

where $a_j = -3, b_j = 3, j = 1, \dots, d$ and $\mathbf{k} \in \mathbb{Z}^d$. After some careful calculations, we have

$$I_1(\mathbf{x}) = \sum_{\mathbf{k}} \hat{\rho}_{\mathbf{k}} \left(\sum_{q=0}^Q w_q G_{\mathbf{k}}^q \right) \prod_{j=1}^d e^{\frac{2\pi i k_j}{b_j - a_j} (x^{(j)} - a_j)}, \quad (2.9)$$

where

$$G_{\mathbf{k}}^q = \prod_{j=1}^d \int_{-2}^2 e^{-\tau_q^2 |y^{(j)}|^2} e^{\frac{-2\pi i k_j y^{(j)}}{b_j - a_j}} dy^{(j)} = \prod_{j=1}^d \int_0^2 2 e^{-\tau_q^2 |y^{(j)}|^2} \cos\left(\frac{2\pi k_j y^{(j)}}{b_j - a_j}\right) dy^{(j)}, \quad (2.10)$$

can be pre-computed once for all if the potential is computed on the same grid.

For the *near-field correction integral* I_2 , within the small ball \mathcal{B}_δ , the density function $\rho_{\mathbf{x}}(\mathbf{y}) := \rho(\mathbf{x} - \mathbf{y})$ is approximated by a low-order Taylor expansion as follows

$$\rho_{\mathbf{x}}(\mathbf{y}) \approx P_{\mathbf{x}}(\mathbf{y}) = \rho_{\mathbf{x}}(\mathbf{0}) + \sum_{j=1}^d \frac{\partial \rho_{\mathbf{x}}(\mathbf{0})}{\partial y_j} y_j + \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2 \rho_{\mathbf{x}}(\mathbf{0})}{\partial y_j \partial y_k} y_j y_k + \frac{1}{6} \sum_{j,k,\ell=1}^d \frac{\partial^3 \rho_{\mathbf{x}}(\mathbf{0})}{\partial y_j \partial y_k \partial y_\ell} y_j y_k y_\ell. \quad (2.11)$$

Next, we integrate in spherical/polar coordinates. The computation boils down to a multiplication of the Laplacian $\Delta \rho$ since the contributions of the odd derivatives in (2.11) and off-diagonal components of the Hessian vanish. The derivatives of ρ are computed by using the Fourier series approximation of the density.

The GauSum method achieves a spectral accuracy and is essentially as efficient as FFT algorithms within $O(N \log N)$ arithmetic operations. The algorithm was implemented for the Coulomb-type kernels in [33]. The evaluations of 2D and 3D DDIs boil down to the Coulomb potentials with some modified densities. More explicitly, the 2D and 3D DDIs can be reformulated as follows

$$\Phi(\mathbf{x}) = -\frac{3}{2} (\partial_{\mathbf{n}_\perp \mathbf{n}_\perp} - n_3^2 \nabla_\perp^2) \left(\frac{1}{2\pi|\mathbf{x}|} \right) * \rho = \left(\frac{1}{2\pi|\mathbf{x}|} \right) * \left(-\frac{3}{2} (\partial_{\mathbf{n}_\perp \mathbf{n}_\perp} \rho - n_3^2 \nabla_\perp^2 \rho) \right), \quad \mathbf{x} \in \mathbb{R}^2, \quad (2.12)$$

$$\Phi(\mathbf{x}) = -(\mathbf{n} \cdot \mathbf{n}) \rho(\mathbf{x}) + \partial_{\mathbf{nn}} \left(\frac{1}{4\pi|\mathbf{x}|} \right) * \rho = -(\mathbf{n} \cdot \mathbf{n}) \rho(\mathbf{x}) + \frac{1}{4\pi|\mathbf{x}|} * (\partial_{\mathbf{nn}} \rho), \quad \mathbf{x} \in \mathbb{R}^3. \quad (2.13)$$

Then, we need to substitute the modified densities, i.e. $\frac{3}{2}(\partial_{\mathbf{n}_\perp \mathbf{n}_\perp} \rho - n_3^2 \nabla_\perp^2 \rho)$ and $(\partial_{\mathbf{nn}} \rho)$ for ρ in (2.1) for the 2D and 3D cases, respectively.

3. Ground state computation: properties, numerical scheme and simulations

In this section, we first prove some results related to the existence/non-existence of the ground states (subsection 3.1). We next propose in subsection 3.2 an efficient and accurate numerical method for computing the ground states by combining the normalized gradient flow which is discretized by the semi-implicit backward Euler Fourier pseudo-spectral method and the Gaussian-Sum nonlocal interaction solver. We shall refer to this new method as *GF-GauSum* hereafter. Finally, subsection 3.3 reports some simulations of the ground states to show some special features related to FNLSEs.

3.1. Existence and nonexistence of the ground states

To simplify the presentation, we divide the energy functional $\mathcal{E}(\phi)$ (1.7) into five parts, i.e. the kinetic, potential, rotating, local and nonlocal interactions energy parts

$$\mathcal{E}(\phi(\mathbf{x})) = \mathcal{E}_{\text{kin}}(\phi) + \mathcal{E}_{\text{pot}}(\phi) + \mathcal{E}_{\text{rot}}(\phi) + \mathcal{E}_{\text{int}}(\phi) + \mathcal{E}_{\text{non}}(\phi), \quad (3.1)$$

where

$$\begin{aligned} \mathcal{E}_{\text{kin}}(\phi) &:= \frac{1}{2} \langle (-\nabla^2 + m^2)^s \phi, \phi \rangle, & \mathcal{E}_{\text{pot}}(\phi) &:= \langle V(\mathbf{x}) \phi, \phi \rangle, \\ \mathcal{E}_{\text{rot}}(\phi) &= -\Omega \langle L_z \phi, \phi \rangle, & \mathcal{E}_{\text{int}}(\phi) &:= \frac{\beta}{2} \langle |\phi|^2, |\phi|^2 \rangle, & \mathcal{E}_{\text{non}}(\phi) &:= \frac{\lambda}{2} \langle \Phi, |\phi|^2 \rangle, \end{aligned}$$

with $\langle f, g \rangle = \int_{\mathbb{R}^d} f \bar{g} d\mathbf{x}$. We first prove some properties of the energy functional $\mathcal{E}(\phi(\mathbf{x}))$ for any $\phi \in S$.

Lemma 3.1. *If the convolution kernel $\mathcal{U}(\mathbf{x})$ in (1.5) is chosen as the Coulomb-type interaction and V is the harmonic potential defined by (1.4), we have the following properties*

(i) *For any positive $\varepsilon > 0$, we have for $\phi \in S$*

$$\left| \langle \Phi, \rho \rangle \right| = \left| \langle \mathcal{U}(\mathbf{x}) * \rho, \rho \rangle \right| \leq \varepsilon \|\nabla \phi\|_2^2 + C_\varepsilon, \quad (3.2)$$

where C_ε is a real-valued constant that depends only on d, μ and ε .

(ii) *When $s > 1$, for any $m \geq 0$ and $\phi \in S$, we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \left[\frac{1}{8} \bar{\phi} (-\nabla^2 + m^2)^s \phi + \left(V(\mathbf{x}) - \frac{\gamma_r^2 |\mathbf{x}|^2}{2} \right) |\phi|^2 + \frac{\beta}{2} |\phi|^4 \right] d\mathbf{x} + C_1 &\leq \mathcal{E}(\phi) \\ &\leq \int_{\mathbb{R}^d} \left[\frac{7}{8} \bar{\phi} (-\nabla^2 + m^2)^s \phi + \left(V(\mathbf{x}) + \frac{\gamma_r^2 |\mathbf{x}|^2}{2} \right) |\phi|^2 + \frac{\beta}{2} |\phi|^4 \right] d\mathbf{x} + C_2, \end{aligned} \quad (3.3)$$

where $\gamma_r = \min\{\gamma_x, \gamma_y\}$, C_1 and C_2 are two constants that only depend on Ω, s, γ_r, d and μ .

Proof. (i) Using the Hardy-Littlewood-Sobolev (HLS) inequality, we have for the Coulomb-type interaction

$$\left| \langle \Phi, \rho \rangle \right| = \left| \langle \mathcal{U}(\mathbf{x}) * \rho, \rho \rangle \right| = \frac{1}{2^{d-1}\pi} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\rho(\mathbf{x}) \rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^\mu} d\mathbf{x} d\mathbf{y} \leq c_{d,\mu} \|\rho\|_p^2 = c_{d,\mu} \|\phi\|_{2p}^4, \quad (3.4)$$

where $1 < p = \frac{2d}{2d-\mu} \leq \frac{2d}{d+1} < 2$ and the constant $c_{d,\mu}$ depends only on d and μ . For the 3D case, let us introduce $\sigma = \frac{3-p}{2p}$. By the Hölder's inequality, Young's inequality and the embedding theorem, we obtain

$$\left| \langle \Phi, \rho \rangle \right| \leq c_{d,\mu} \left(\|\phi\|_2^\sigma \|\phi\|_6^{1-\sigma} \right)^4 = c_{d,\mu} \left(\|\phi\|_6^2 \right)^{2(1-\sigma)} \leq \tilde{\varepsilon} \|\phi\|_6^2 + C_\varepsilon \leq \varepsilon \|\nabla \phi\|_2^2 + C_\varepsilon. \quad (3.5)$$

Similarly, for the 2D case, let $q = \frac{4p}{2-p} > 2p > 2$ and $\sigma = \frac{q-2p}{p(q-2)}$. Then, one gets

$$\left| \langle \Phi, \rho \rangle \right| \leq c_{d,\mu} \left(\|\phi\|_2^\sigma \|\phi\|_q^{1-\sigma} \right)^4 = c_{d,\mu} \left(\|\phi\|_q^2 \right)^{2(1-\sigma)} \leq \tilde{\varepsilon} \|\phi\|_q^2 + \widetilde{C}_\varepsilon \leq \varepsilon \|\nabla \phi\|_2^2 + C_\varepsilon. \quad (3.6)$$

(ii) Let $\gamma_r = \min\{\gamma_x, \gamma_y\}$. By Young's inequality and Plancherel's formula, we have

$$\begin{aligned} \left| \Omega \int_{\mathbb{R}^d} \bar{\phi} L_z \phi \, d\mathbf{x} \right| &\leq \int_{\mathbb{R}^d} \left[|(\gamma_r x \bar{\phi})(\Omega \partial_y \phi / \gamma_r)| + |(\gamma_r y \bar{\phi})(\Omega \partial_x \phi / \gamma_r)| \right] d\mathbf{x} \\ &\leq \frac{\gamma_r^2}{2} \int_{\mathbb{R}^d} |\mathbf{x}|^2 |\phi|^2 d\mathbf{x} + \frac{\Omega^2}{2\gamma_r^2} \int_{\mathbb{R}^d} |\nabla \phi|^2 d\mathbf{x} = \frac{\gamma_r^2}{2} \int_{\mathbb{R}^d} |\mathbf{x}|^2 |\phi|^2 d\mathbf{x} + \frac{\Omega^2}{2\gamma_r^2 (2\pi)^d} \int_{\mathbb{R}^d} |\mathbf{k}|^2 |\widehat{\phi}|^2 d\mathbf{k} \\ &\leq \frac{\gamma_r^2}{2} \int_{\mathbb{R}^d} |\mathbf{x}|^2 |\phi|^2 d\mathbf{x} + \frac{\Omega^2}{2\gamma_r^2 (2\pi)^d} \int_{\mathbb{R}^d} \left[\frac{\gamma_r^2 (|\mathbf{k}|^2 + m^2)^s}{2\Omega^2} + \left(\frac{\gamma_r^2}{2\Omega^2} \right)^{\frac{1}{1-s}} \right] |\widehat{\phi}|^2 d\mathbf{k} - \frac{\Omega^2 m^2}{2\gamma_r^2} \\ &\leq \frac{\gamma_r^2}{2} \int_{\mathbb{R}^d} |\mathbf{x}|^2 |\phi|^2 d\mathbf{x} + \frac{1}{4} \int_{\mathbb{R}^d} \bar{\phi} (-\nabla^2 + m^2)^s \phi \, d\mathbf{x} + C. \end{aligned} \quad (3.7)$$

Similarly, for the Coulomb-type nonlocal interaction, we obtain

$$\begin{aligned} \left| \frac{\lambda}{2} \langle \Phi, \rho \rangle \right| &\leq \tilde{\varepsilon} \|\nabla \phi\|_2^2 + \widetilde{C}_\varepsilon = \frac{\tilde{\varepsilon}}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathbf{k}|^2 |\widehat{\phi}|^2 d\mathbf{k} + \widetilde{C}_\varepsilon \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left[\frac{1}{8} (|\mathbf{k}|^2 + m^2)^s + C \right] |\widehat{\phi}|^2 d\mathbf{k} \\ &= \frac{1}{8} \int_{\mathbb{R}^d} \bar{\phi} (-\nabla^2 + m^2)^s \phi \, d\mathbf{x} + C. \end{aligned} \quad (3.8)$$

Therefore, the inequality (3.3) follows from (3.7) and (3.8). \square

Theorem 3.1. *If $V(\mathbf{x})$ is a trapping harmonic potential defined in (1.4), then the following properties hold.*

(i) *If $s > 1$ and $\beta \geq 0$, then there exists a ground state of the FNLSE for all $\Omega > 0$ if one of the following conditions holds:*

(A) *$\mathcal{U}(\mathbf{x})$ reads as either Coulomb-type.*

(B) *For 3D DDI: $-\beta/2 \leq \lambda \leq \beta$.*

(C) *For 2D DDI: (c1) $\lambda = 0$. (c2) $\lambda > 0$ and $n_3 = 0$. (c3) $\lambda < 0$ and $n_3^2 \geq \frac{1}{2}$.*

(ii) *If $\Omega = 0$, $\beta > 0$ and $\lambda > 0$, then the ground state of the FNLSE exists for all $s > 0$.*

(iii) *If $0 < s < 1$, there exists no ground state if one of the following conditions holds*

(A) *$\forall \Omega > 0$, $\mathcal{U}(\mathbf{x})$ is a Coulomb-type interaction or a 3D DDI.*

(B) *$\mathcal{U}(\mathbf{x})$ is the 2D DDI, $\forall \Omega > \Omega_0 = c|\lambda|^{\frac{2}{5}}$ with $c = \left(\frac{(2\pi^2+1)^4 \gamma^6}{48e\pi^9} \right)^{\frac{1}{5}} (\approx 0.54 \text{ for } \gamma = 1)$. Here, $\gamma = \max\{\gamma_x, \gamma_y\}$*

Proof: (i) For the Coulomb-type interaction, it is clear by Lemma 3.1 that the energy functional \mathcal{E} is bounded below, coercive and weakly lower semi-continuous on S . Hence, (A) follows. For the DDI, the proof is similar as those for the non-fractional case [10, 12] by noticing (3.7). Similar arguments lead to (ii).

(iii) Denote $\gamma = \max\{\gamma_x, \gamma_y\}$. In 2D, we choose the function

$$\phi_n(\mathbf{x}) = \mathcal{F}^{-1}(\widehat{\phi}_n)(\mathbf{x}), \quad \text{with} \quad \widehat{\phi}_n(\mathbf{k}) = \mathcal{F}(\phi_n)(\mathbf{k}) = (4\pi\varepsilon^{n+1})^{1/2} (n!)^{-1/2} \exp(-\varepsilon|\mathbf{k}|^2/2) |\mathbf{k}|^n e^{in\theta}. \quad (3.9)$$

By Plancherel's formula, it is easy to check that $\|\phi_n\|_2 = \frac{1}{2\pi}\|\widehat{\phi_n}\|_2 = 1$, and thus $\phi_n \in S$. Let $\rho_n = |\phi_n|^2$. By Young's inequality and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathcal{E}_1(\phi_n) &= \mathcal{E}_{\text{kin}}(\phi_n) + \mathcal{E}_{\text{pot}}(\phi_n) + \mathcal{E}_{\text{rot}}(\phi_n) \\ &\leq \frac{1}{4\pi^2} \left[\frac{1}{2} \langle (|\mathbf{k}|^2 + m^2)^s \widehat{\phi_n}, \widehat{\phi_n} \rangle - \frac{\gamma^2}{2} \langle \Delta \widehat{\phi_n}, \widehat{\phi_n} \rangle - i\Omega \langle \widehat{J}_{z_k} \widehat{\phi_n}, \widehat{\phi_n} \rangle \right] \\ &\leq \frac{e^{m^2\varepsilon}}{2\varepsilon^s} \frac{\Gamma(n+1+s)}{\Gamma(n+1)} + \left(\frac{\varepsilon\gamma^2}{2} - \Omega \right) n + \frac{\varepsilon\gamma^2}{2},\end{aligned}\tag{3.10}$$

$$\begin{aligned}|\mathcal{E}_{\text{int}}(\phi_n)| &= \frac{|\beta|}{2} \|\rho_n\|_2^2 = \frac{|\beta|}{2(2\pi)^6} \|\widehat{\phi_n} * \widehat{\phi_n}^*\|_2^2 \leq \frac{|\beta|}{2(2\pi)^4} \left(\frac{1}{(2\pi)^2} \|\widehat{\phi_n}\|_2^2 \right) \|\widehat{\phi_n}\|_1^2 \\ &= \frac{|\beta|}{2(2\pi)^4} \|\widehat{\phi_n}\|_1^2 = \frac{|\beta|}{2\pi\varepsilon} \frac{2^n (\Gamma(n/2+1))^2}{\Gamma(n+1)}.\end{aligned}\tag{3.11}$$

Furthermore, we compute the nonlocal interaction energy $\mathcal{E}_{\text{non}}(\phi_n)$. For the Coulomb-type interaction, by using the HLS inequality (3.4) and the Hölder's inequality, we obtain

$$|\mathcal{E}_{\text{non}}(\phi_n)| \leq \frac{c_\mu |\lambda|}{2} \|\rho_n\|_p^2 \leq \tilde{c}_\mu \|\rho_n\|_1^{2-\mu} \|\rho_n\|_2^\mu \leq \tilde{c}_\mu \left[\frac{1}{\pi\varepsilon} \frac{2^n (\Gamma(n/2+1))^2}{\Gamma(n+1)} \right]^{\mu/2},\tag{3.12}$$

where $p = \frac{4}{4-\mu}$, $0 < \mu \leq 1$, and \tilde{c}_μ depends only on μ and λ . Together with the Stirling's formula

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} \right)^x, \text{ when } x \rightarrow \infty,\tag{3.13}$$

one gets

$$|\mathcal{E}_{\text{int}}(\phi_n)| \sim \sqrt{n}, \quad |\mathcal{E}_{\text{non}}(\phi_n)| \sim \sqrt{n}, \quad n \rightarrow \infty.\tag{3.14}$$

Let $\varepsilon < \frac{2\Omega}{\gamma^2}$, $\forall \Omega > 0$ and $s < 1$, we then prove that

$$\begin{aligned}\limsup_{n \rightarrow \infty} \mathcal{E}(\phi_n) &\leq \limsup_{n \rightarrow \infty} [\mathcal{E}_1(\phi_n) + |\mathcal{E}_{\text{int}}(\phi_n)| + \mathcal{E}_{\text{non}}(\phi_n)] \\ &\leq \limsup_{n \rightarrow \infty} \left[\frac{c_2 n^s}{2\varepsilon^s} + \left(\frac{\varepsilon\gamma^2}{2} - \Omega \right) n + c_1 \sqrt{n} + c_0 \right] = -\infty,\end{aligned}\tag{3.15}$$

which implies the nonexistence of the ground states.

For the 2D DDI, we have

$$|\mathcal{E}_{\text{non}}(\phi_n)| = \frac{|\lambda|}{8\pi^2} |\langle \widehat{\mathcal{U}}_{\text{ker}} \widehat{\rho_n}, \widehat{\rho_n} \rangle| \leq \frac{|\lambda|}{8\pi^2} \left\langle \frac{3(|\mathbf{k} \cdot \mathbf{n}_\perp|^2 + n_3^2 |\mathbf{k}|^2)}{2|\mathbf{k}|} \widehat{\rho_n}, \widehat{\rho_n} \right\rangle \leq \frac{3|\lambda|}{16\pi^2} \langle |\mathbf{k}| \widehat{\rho_n}, \widehat{\rho_n} \rangle.\tag{3.16}$$

By the generalized Minkowski inequality, we prove that

$$\begin{aligned}4\pi^2 \left(\langle |\mathbf{k}| \widehat{\rho_n}, \widehat{\rho_n} \rangle \right)^{\frac{1}{2}} &= \left[\int_{\mathbb{R}^2} |\mathbf{k}| \left| \int_{\mathbb{R}^2} \widehat{\phi_n}(\mathbf{k} - \boldsymbol{\xi}) \widehat{\phi_n}(\boldsymbol{\xi}) d\boldsymbol{\xi} \right|^2 d\mathbf{k} \right]^{\frac{1}{2}} \leq \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} |\mathbf{k}| |\widehat{\phi_n}(\mathbf{k} - \boldsymbol{\xi})|^2 |\widehat{\phi_n}(\boldsymbol{\xi})|^2 d\mathbf{k} \right]^{\frac{1}{2}} d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^2} |\widehat{\phi_n}(\boldsymbol{\xi})| \left[\int_{\mathbb{R}^2} |\mathbf{k} - \boldsymbol{\xi}| |\widehat{\phi_n}(\mathbf{k})|^2 d\mathbf{k} \right]^{\frac{1}{2}} d\boldsymbol{\xi} \leq \int_{\mathbb{R}^2} |\widehat{\phi_n}(\boldsymbol{\xi})| \left[\int_{\mathbb{R}^2} (|\mathbf{k}| + |\boldsymbol{\xi}|) |\widehat{\phi_n}(\mathbf{k})|^2 d\mathbf{k} \right]^{\frac{1}{2}} d\boldsymbol{\xi} \\ &\leq \int_{\mathbb{R}^2} |\widehat{\phi_n}(\boldsymbol{\xi})| \left[\sqrt{|\boldsymbol{\xi}|} + \left(\int_{\mathbb{R}^2} |\mathbf{k}| |\widehat{\phi_n}(\mathbf{k})|^2 d\mathbf{k} \right)^{\frac{1}{2}} \right] d\boldsymbol{\xi} = \left(\|\sqrt{|\mathbf{k}|} \widehat{\phi_n}\|_1 + \|\widehat{\phi_n}\|_1 \|\sqrt{|\mathbf{k}|} \widehat{\phi_n}\|_2 \right) \\ &= \frac{\pi^{\frac{1}{2}} 2^{\frac{n}{2} + \frac{9}{4}}}{\varepsilon^{\frac{3}{4}}} \frac{\Gamma(\frac{n}{2} + \frac{5}{4})}{\sqrt{\Gamma(n+1)}} + \frac{\pi^{\frac{5}{2}} 2^{\frac{n}{2} + 3}}{\varepsilon^{\frac{3}{4}}} \frac{\sqrt{\Gamma(n + \frac{3}{2})} \Gamma(\frac{n}{2} + 1)}{\Gamma(n+1)}.\end{aligned}\tag{3.17}$$

Again, by the Stirling's formula (3.13), we show that

$$|\mathcal{E}_{\text{non}}(\phi_n)| \lesssim \frac{c_3 |\lambda|}{\varepsilon^{3/2}} n, \quad n \rightarrow \infty, \quad (3.18)$$

where $c_3 = \frac{3\sqrt{2}(2\pi^2+1)^2}{32\pi\sqrt[3]{e\pi}}$. Let us set $\varepsilon = \left(3c_3|\lambda|/\gamma^2\right)^{\frac{2}{5}}$ and $\Omega > \Omega_0 = \left(\frac{(2\pi^2+1)^4\gamma^6}{48e\pi^9}\right)^{\frac{1}{5}}|\lambda|^{\frac{2}{5}}$. It follows that

$$\limsup_{n \rightarrow \infty} \mathcal{E}(\phi_n) \leq \limsup_{n \rightarrow \infty} \left[\frac{c_2 n^s}{2\varepsilon^s} + \left(\frac{\varepsilon\gamma^2}{2} + \frac{c_3}{\varepsilon^{\frac{3}{2}}} - \Omega \right) n + c_1 \sqrt{n} + c_0 \right] = -\infty,$$

leading to the nonexistence of the ground states.

In 3D, we choose the sequence

$$\phi_n^{\text{3D}}(\mathbf{x}) = \phi_n(x, y)\phi(z), \quad (3.19)$$

where $\phi(z) = \left(\frac{\gamma_z}{\pi}\right)^{1/4} \exp\{-\frac{\gamma_z z^2}{2}\}$ and $\phi_n(x, y) = \mathcal{F}^{-1}(\hat{\phi}_n(\mathbf{k}))$, with $\hat{\phi}_n(\mathbf{k})$ reading as (3.9). Then, the argument proceeds similarly as those in 2D for the 3D Coulomb potential. As for the 3D DDI, noticing that

$$|\mathcal{E}_{\text{non}}(\phi_n^{\text{3D}})| \leq \frac{3|\lambda|}{2} \|\phi_n^{\text{3D}}\|_4^4 = \frac{3|\lambda|\sqrt{\gamma_z}}{2\sqrt{2\pi}} \|\phi_n\|_4^4 = \frac{3|\lambda|\sqrt{\gamma_z}}{2\sqrt{2\pi}} \|\rho_n\|_2^2, \quad (3.20)$$

the left argument proceeds similarly as those in 2D from (3.11). \square

Remark 3.1. For the 2D DDI, one open question concerns the plausible fact that (iii)(B) in Theorem (3.1) maybe hold for $\forall \Omega_0 > 0$. The proof presented here does not seem to be directly applicable for this conjecture.

Remark 3.2. It might be interesting to understand the existence/non-existence and the uniqueness of the ground states for the more general FNLSE

$$i\partial_t \psi = \left[\frac{1}{2}(-\nabla^2 + m^2)^s + \frac{1}{2}\gamma_r^2 |\mathbf{x}|^p + \beta |\psi|^q + \lambda \Phi - \Omega L_z \right] \psi, \quad (3.21)$$

where the constants β and λ can be positive or negative and the powers p and q are real-valued positive constants. We leave it as an open problem for some future studies.

3.2. Numerical method

For a constant time step Δt , we introduce the discrete times $t_n = n\Delta t$ for $n = 0, 1, 2, \dots$. The gradient flow with discrete normalization (GFDN) method reads as

$$\partial_t \phi(\mathbf{x}, t) = - \left[\frac{1}{2}(-\nabla^2 + m^2)^s + V(\mathbf{x}) + \beta |\phi|^2 + \lambda \Phi(\mathbf{x}, t) - \Omega L_z \right] \phi(\mathbf{x}, t), \quad (3.22)$$

$$\Phi(\mathbf{x}, t) = (\mathcal{U} * |\phi|^2)(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^d, \quad t_n \leq t < t_{n+1}, \quad (3.23)$$

$$\phi(\mathbf{x}, t_{n+1}) = \frac{\phi(\mathbf{x}, t_{n+1}^-)}{\|\phi(\mathbf{x}, t_{n+1}^-)\|_2}, \quad \mathbf{x} \in \mathbb{R}^d, \quad n \geq 0, \quad (3.24)$$

with the initial data

$$\phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad \text{with} \quad \|\phi_0\|_2 = 1. \quad (3.25)$$

Let $\phi^n(\mathbf{x})$ and $\Phi^n(\mathbf{x})$ be the approximations of $\phi(\mathbf{x}, t_n)$ and $\Phi(\mathbf{x}, t_n)$, respectively. The above GFDN is usually discretized in time *via* the semi-implicit backward Euler method [5, 15, 17, 80]

$$\frac{\phi^{(1)}(\mathbf{x}) - \phi^n(\mathbf{x})}{\Delta t} = - \left[\frac{1}{2}(-\nabla^2 + m^2)^s + V(\mathbf{x}) + \beta |\phi^n|^2 + \lambda \Phi^n(\mathbf{x}) - \Omega L_z \right] \phi^{(1)}(\mathbf{x}), \quad (3.26)$$

$$\Phi^n(\mathbf{x}) = (\mathcal{U} * |\phi^n|^2)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad (3.27)$$

$$\phi^{n+1}(\mathbf{x}) = \frac{\phi^{(1)}(\mathbf{x})}{\|\phi^{(1)}(\mathbf{x})\|_2}, \quad \mathbf{x} \in \mathbb{R}^d, \quad n \geq 0. \quad (3.28)$$

The ground states decay exponentially fast due to the trapping potential. Therefore, in practical computations, we first truncate the whole space to a bounded rectangular domain and impose periodic boundary conditions. Then, we discretize the equation (3.26) *via* the Fourier pseudo-spectral method in space and evaluate the nonlocal interaction $\Phi^n(\mathbf{x})$ by the GauSum solver. The full discretized scheme of system (3.26)-(3.28) can then be solved by a fixed-point iteration or a preconditioned Krylov subspace solver with a similar preconditioner as those in [5]. Let us define the operators

$$A^{\text{BE},n} := \frac{I}{\Delta t} + \frac{1}{2}(-\nabla^2 + m^2)^s + V(\mathbf{x}) + \beta|\phi^n|^2 + \lambda \Phi^n(\mathbf{x}) - \Omega L_z, \quad (3.29)$$

$$P_{\Delta}^{\text{BE}} = \left[\frac{I}{\Delta t} + \frac{1}{2}(-\nabla^2 + m^2)^s \right]^{-1}, \quad A_{\text{TF}}^{\text{BE},n} = V(\mathbf{x}) + \beta|\phi^n|^2 + \lambda \Phi^n(\mathbf{x}) - \Omega L_z, \quad (3.30)$$

$$P_{\text{TF}}^{\text{BE},n} = \left[\frac{I}{\Delta t} + V(\mathbf{x}) + \beta|\phi^n|^2 + \lambda \Phi^n(\mathbf{x}) \right]^{-1}, \quad A_{\Delta,\Omega}^{\text{BE},n} = \frac{1}{2}(-\nabla^2 + m^2)^s - \Omega L_z. \quad (3.31)$$

Moreover, we denote by \mathbb{I} , $\mathbb{A}^{\text{BE},n}$, $\mathbb{P}_{\Delta}^{\text{BE}}$, $\mathbb{A}_{\text{TF}}^{\text{BE},n}$, $\mathbb{P}_{\text{TF}}^{\text{BE},n}$, $\mathbb{A}_{\Delta,\Omega}^{\text{BE},n}$ the discretized versions of the above operators, and by $\phi^{(1)}$ and ϕ^n the discretization of $\phi^{(1)}$ and ϕ^n through the Fourier pseudo-spectral approximation. Then, the finite-dimensional linear system corresponding to (3.26)-(3.28) reads as

$$\mathbb{A}^{\text{BE},n} \phi^{(1)} = \mathbf{b}^n := \phi^n / \Delta t. \quad (3.32)$$

Two preconditioned versions of the linear system are the following

$$\left(\mathbb{I} + \mathbb{P}_{\Delta}^{\text{BE}} \mathbb{A}_{\text{TF}}^{\text{BE},n} \right) \phi^{(1)} = \mathbb{P}_{\Delta}^{\text{BE}} \mathbf{b}^n, \quad \text{or} \quad \left(\mathbb{I} + \mathbb{P}_{\text{TF}}^{\text{BE},n} \mathbb{A}_{\Delta,\Omega}^{\text{BE},n} \right) \phi^{(1)} = \mathbb{P}_{\text{TF}}^{\text{BE},n} \mathbf{b}^n. \quad (3.33)$$

We refer the reader to [5] for more details and omit them here for brevity. Like in the standard case [5, 8], the most efficient solver uses the first preconditioned system (left) in (3.33) based on $\mathbb{P}_{\Delta}^{\text{BE}}$. In particular, the acceleration of the convergence of the Krylov subspace solver (BiCGStab) is visible when Ω , β and λ are large. In practice, we use this preconditioned solver in subsection 3.3.

3.3. Numerical results

In this subsection, we report some numerical results concerning the ground states of (1.1)-(1.2) computed by the *GF-GauSum* solver built in the previous subsection. To this end, unless stated, we fix $m = 0$ and $d = 2$. We carry out the computation on the domain $\mathbf{B} = [-32, 32] \times [-32, 32]$ that is discretized with uniform mesh sizes $h_x = h_y = \frac{1}{8}$. We use a constant time step $\Delta t = 10^{-3}$. The trapping potential $V(\mathbf{x})$ is chosen as (1.4) with $\gamma_x = \gamma_y = 1$. The nonlocal interaction is of Coulomb-type with $\mu = 1$. The initial guess $\phi_0(\mathbf{x})$ is chosen as

$$\phi_0(\mathbf{x}) = \frac{(1 - \Omega)\phi_{\text{ho}}(\mathbf{x}) + \Omega\phi_{\text{ho}}^v(\mathbf{x})}{\|(1 - \Omega)\phi_{\text{ho}}(\mathbf{x}) + \Omega\phi_{\text{ho}}^v(\mathbf{x})\|}, \quad \text{with} \quad \phi_{\text{ho}}(\mathbf{x}) = \frac{1}{\sqrt{\pi}} e^{-\frac{|\mathbf{x}|^2}{2}}, \quad \phi_{\text{ho}}^v(\mathbf{x}) = \frac{x + iy}{\sqrt{\pi}} e^{-\frac{|\mathbf{x}|^2}{2}}, \quad \mathbf{x} \in \mathbf{B}. \quad (3.34)$$

The ground state $\phi_g(\mathbf{x})$ is reached when the stopping criterion holds: $\|\phi^n(\mathbf{x}) - \phi^{n+1}(\mathbf{x})\|_{\infty} \leq \varepsilon_0 \Delta t$. In the computations, we choose the accuracy parameter $\varepsilon_0 = 10^{-9}$.

Example 3.1. Non-rotating FNLSE. *Here, we impose $\Omega = 0$. We study the ground states of the following four cases:*

- **Case I.** *Linear case, i.e. $\beta = \lambda = 0$.*
- **Case II.** *Purely long-range interaction, i.e. $\beta = 0$ and $\lambda = 10$.*
- **Case III.** *Purely short-range interaction, i.e. $\beta = 10$ and $\lambda = 0$.*
- **Case IV.** *Both long-range and short-range interactions, i.e. $\lambda = \beta = 10$.*

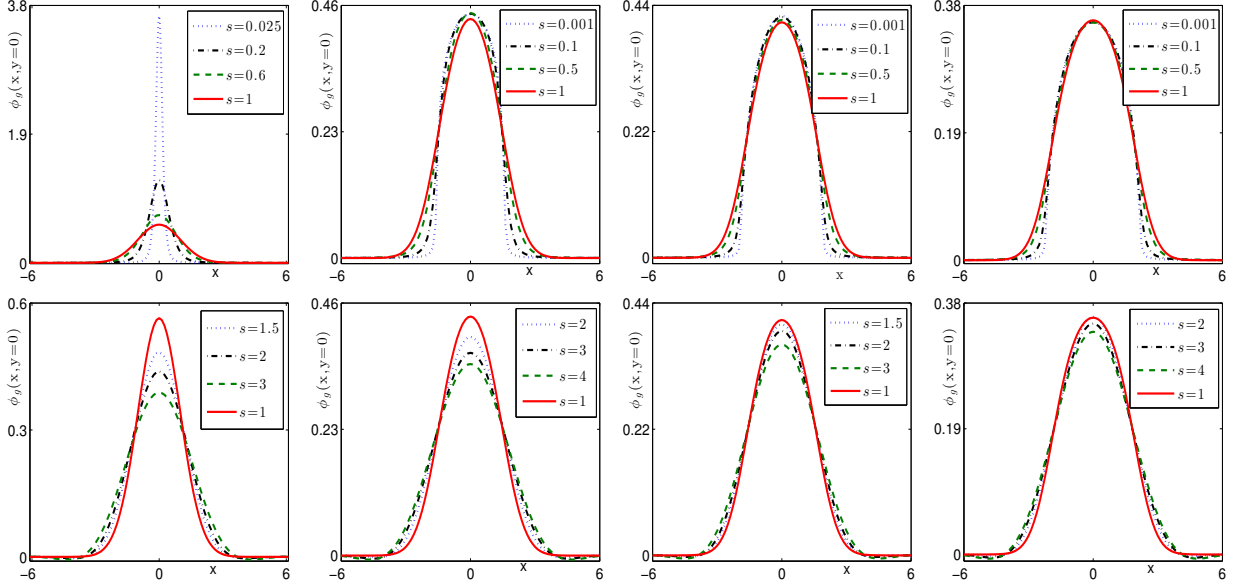


Figure 1: Slice plots of $\phi_g(x, 0)$ for **Cases I–IV** (from left to right) for subdispersion $s \leq 1$ (top row) and superdispersion $s \geq 1$ (bottom) in example 3.1.

Figure 1 shows the slice plots of the ground states along the x -axis, i.e. $\phi_g(x, 0)$, for different fractional orders s of the FNLSE.

Example 3.2. Non-rotating FNLSE with harmonic + optical lattice potential. Here, we choose $\Omega = 0$. We consider the ground states of the FNLSE in a harmonic plus optical lattice potential with different parameters. To this end, we let $\lambda = 64$ and $\beta = 0$ and choose the potential as

$$V(x, y) = \frac{x^2 + y^2}{2} + 10(\sin^2(\pi x) + \sin^2(\pi y)).$$

The spatial mesh sizes are chosen as $h_x = h_y = \frac{1}{32}$ in this case. Figure 2 shows the contour plot of the ground state density $\rho_g := |\phi_g(\mathbf{x})|^2$ and the slice plot of $\phi_g(x, 0)$ with different fractional orders s .

From Figures 1–2 and additional results not shown here, we can conclude that (i) The ground states become more peaked and narrower as the fractional order s tends smaller, which corresponds to subdispersion. (ii) A large fractional order helps in smoothing out the density profile (cf. Fig. 2) for the superdispersion case. (iii) The repulsive local/nonlocal interactions suppress the “focus” or “homogenization” effect as the dispersive order s tends smaller or larger. In other words, the repulsive nonlinear interaction helps to stabilize the ground states. (iv) When β and/or λ are/is large, the nonlinear interaction dominates and the dispersive effect can be neglected.

Example 3.3. Rotating FGPE. In this example, we present the ground states of the rotating FGPE with only local nonlinear interaction, i.e. $\lambda = 0$ and $\beta = 100$.

We propose to numerically study the dependence of the first critical rotating velocity Ω_c to create a vortex with respect to the fractional dispersive order s . Figure 3 shows this relation derived by a linear regression

$$\Omega_c(s) \approx -0.02634 s^2 + 0.19393 s + 0.21071. \quad (3.35)$$

Figure 4 displays the contour plots of the ground state density ρ_g for different values of Ω but with $s = 1.2$ (superdispersion). From Figure 3–4 and additional results not shown here, we can conclude that (i) The first critical rotating velocity Ω_c depends almost linearly on s . (ii) For the superdispersion case, i.e. $s > 1$, the

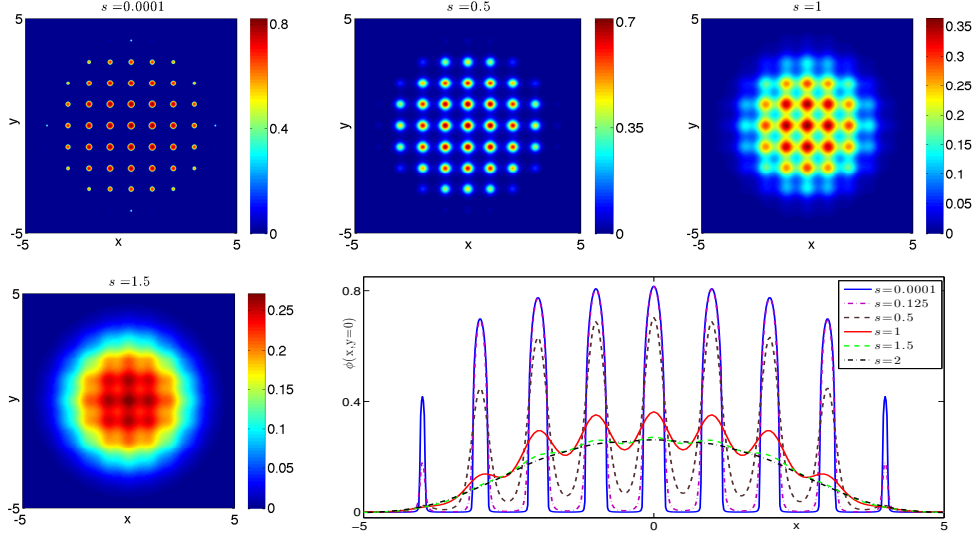


Figure 2: Contour plots of the density of the ground state $\phi_g(x, y)$ and the slice plot of $\phi_g(x, y = 0)$ in example 3.2.

ground states exist for all velocities Ω . As Ω increases, the ground states will undergo three phase transitions (similar to the non-fractional GPE with quartic order trapping potential), i.e., from Gaussian-type to one-vortex profile, from vortex lattice to vortex-lattice with a hole at the center and then to a giant vortex. It would be interesting to study how these critical rotating frequencies for the transitions depend on s and how they compare with those in the case of the standard GPE.

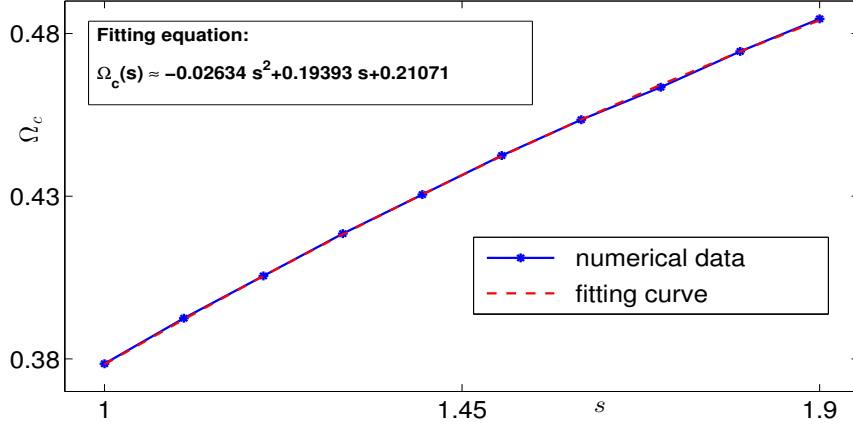


Figure 3: Critical rotating frequency vs. the fractional order s in example 3.3.

4. Dynamics computation: properties, numerical scheme and simulations

In this section, we first present analogous dynamical laws for some commonly used quantities in the classical rotating GPE. Then, we extend the rotating Lagrangian coordinates transform proposed for the standard GPE [16] to the FGPE. In the rotating Lagrangian coordinates, the rotation term vanishes, giving rise to a time-dependent potential. Based on the new FNLSE, we propose a time-splitting Fourier pseudo-spectral method incorporated with the GauSum solver to simulate the dynamics.

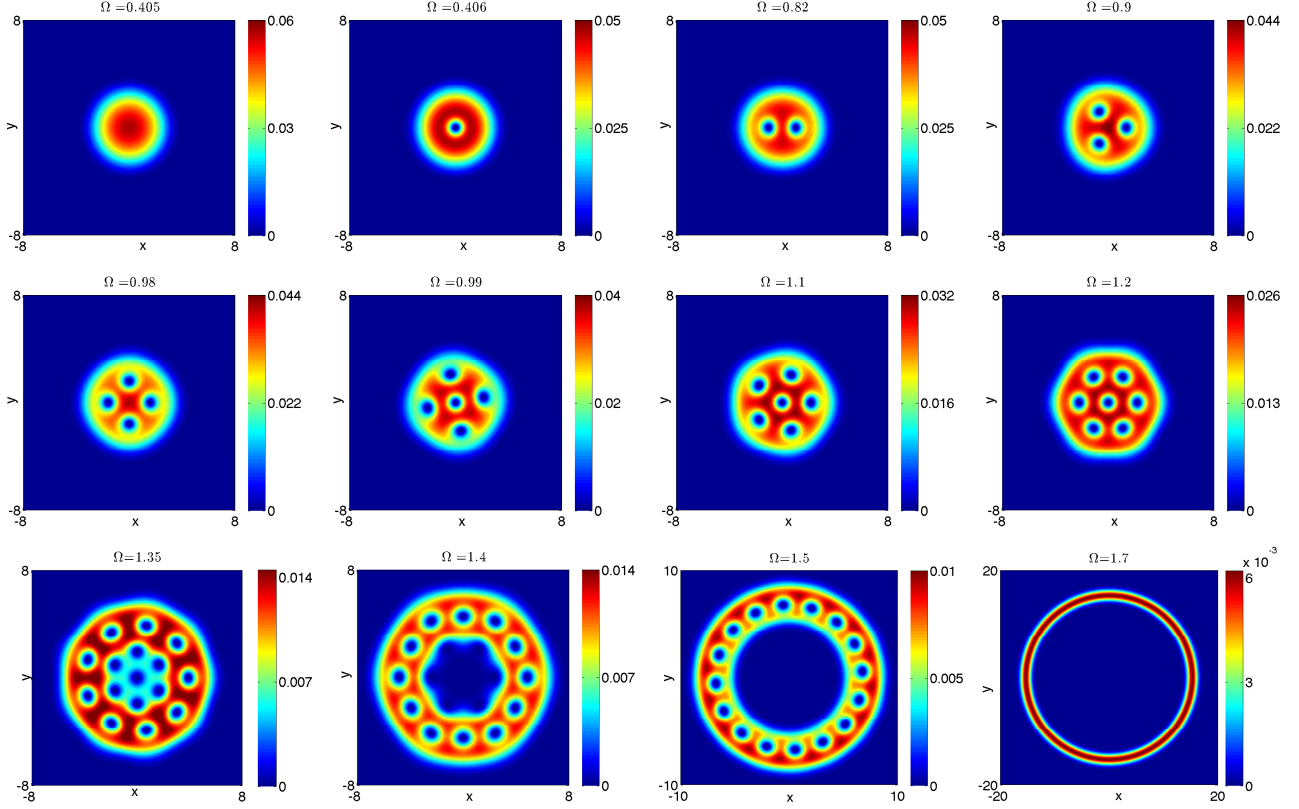


Figure 4: Contour plots of the density $|\phi_g(\mathbf{x})|^2$ in example 3.3 (superdispersion).

4.1. Dynamical properties

Here we study the dynamical properties of the mass, energy, angular momentum expectation and center of mass [16]. The dynamical laws can be used as benchmarks to test the numerical methods and are briefly listed here. For details, one can either refer to appendices or to [71] for analogous proofs to their non-fractional counterparts.

Mass and energy. The FNLSE (1.1)-(1.2) conserves the mass (1.6) and the energy (1.7), i.e.

$$\mathcal{N}(t) = \mathcal{N}(t=0), \quad \mathcal{E}(t) = \mathcal{E}(t=0). \quad (4.1)$$

Proof: It is straightforward to prove in a similar way as their non-fractional counterparts [71] by using the Plancherel's formula. \square

Angular momentum expectation. The *angular momentum expectation* is defined as

$$\langle L_z \rangle(t) = \int_{\mathbb{R}^d} \bar{\psi}(\mathbf{x}, t) L_z \psi(\mathbf{x}, t) d\mathbf{x}, \quad t \geq 0. \quad (4.2)$$

Lemma 4.1. *The angular momentum expectation $\langle L_z \rangle(t)$ satisfies the following equation*

$$\frac{d}{dt} \langle L_z \rangle(t) = \int_{\mathbb{R}^d} |\psi|^2 (y \partial_x - x \partial_y) (V(\mathbf{x}) + \lambda \Phi(\mathbf{x}, t)) d\mathbf{x}. \quad (4.3)$$

This implies that the angular momentum expectation is conserved, i.e.

$$\langle L_z \rangle(t) = \langle L_z \rangle(0), \quad t \geq 0, \quad (4.4)$$

when $V(\mathbf{x})$ is radially/cylindrically symmetric in 2D/3D and one of the following conditions holds: (i) $\lambda = 0$, (ii) $\lambda \neq 0$, $\Phi(\mathbf{x})$ is the Coulomb potential or (iii) $\lambda \neq 0$, $\Phi(\mathbf{x})$ is the dipole potential with dipole axis $\mathbf{n} = (0, 0, 1)^T$, i.e. is parallel to the z -axis.

Proof. Details of the proof are given in Appendix A. \square

Center of mass. The center of mass is defined by

$$\mathbf{x}_c(t) = \int_{\mathbb{R}^d} \mathbf{x} |\psi(\mathbf{x}, t)|^2 d\mathbf{x} = \langle \mathbf{x} \psi, \psi \rangle. \quad (4.5)$$

Lemma 4.2. The center of mass $\mathbf{x}_c(t)$ satisfies the following equations, for $0 < s \leq 1$ (subdispersion),

$$\dot{\mathbf{x}}_c - \Omega J \mathbf{x}_c = i \langle G * \psi, \nabla \psi \rangle, \quad (4.6)$$

$$\ddot{\mathbf{x}}_c - 2\Omega J \dot{\mathbf{x}}_c + \Omega^2 J^2 \mathbf{x}_c = 2\text{Re} \left(\langle G * (\mathcal{V}\psi), \nabla \psi \rangle \right). \quad (4.7)$$

Here, we set $\mathcal{V}(\mathbf{x}, |\psi|) = V(\mathbf{x}) + \beta |\psi|^2 + \lambda \Phi(\mathbf{x}, t)$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } d = 2, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{for } d = 3. \quad (4.8)$$

The convolution kernel $G(\mathbf{x})$ reads as

$$G(\mathbf{x}) = \begin{cases} \delta(\mathbf{x}), & s = 1, \\ \frac{2^{s-d/2} s}{\Gamma(1-s) \pi^{d/2}} \left(\frac{m}{|\mathbf{x}|} \right)^{\frac{d}{2}+s-1} K_{\frac{d}{2}+s-1}(m|\mathbf{x}|), & 0 < s < 1, \end{cases} \quad (4.9)$$

where $\delta(\mathbf{x})$ is the Dirac delta function and $K_v(z)$, the modified Bessel function of the second-kind and order v , is given explicitly as follows

$$K_v(z) = \frac{(2z)^v \Gamma(v + \frac{1}{2})}{\sqrt{\pi}} \int_0^\infty \frac{\cos(t)}{(t^2 + z^2)^{v+\frac{1}{2}}} dt. \quad (4.10)$$

Proof. A detailed proof is reported in Appendix B. \square

Remark 4.1. If $s = 1$, $V(\mathbf{x})$ is the harmonic potential (1.4) and $\Phi(\mathbf{x})$ is the Coulomb potential or DDI with $\mathbf{n} = (0, 0, 1)^T$, then (4.7) reduces to [16, 51]

$$\ddot{\mathbf{x}}_c - 2\Omega J \dot{\mathbf{x}}_c + (\Omega^2 J^2 + \Lambda_d) \mathbf{x}_c = \mathbf{0}, \quad (4.11)$$

where

$$\Lambda_d = \begin{pmatrix} \gamma_x^2 & 0 \\ 0 & \gamma_y^2 \end{pmatrix}, \quad \text{for } d = 2, \quad \Lambda_d = \begin{pmatrix} \Lambda_2 & \mathbf{0} \\ \mathbf{0} & \gamma_z^2 \end{pmatrix}, \quad \text{for } d = 3. \quad (4.12)$$

In [51], the authors derived a dynamical law for the center of mass for the FNLSE with $s \in (\frac{1}{2}, 1]$ and for a harmonic trapping potential. Compared with their results, the dynamical laws (4.6)-(4.7) are simpler and hold for a general potential $V(\mathbf{x})$ as well as for the full subdispersion case, i.e. $\forall s \in (0, 1]$. It is also interesting to explore similar equations for the superdispersion case $s > 1$.

Remark 4.2. We also remark here that it might be interesting to derive the dynamical laws for the condensate width δ_v which is defined as

$$\delta_v(t) = \int_{\mathbb{R}^d} v^2 |\psi(\mathbf{x})|^2 d\mathbf{x}, \quad v = x, y \text{ in 2D and } v = x, y, z \text{ in 3D}. \quad (4.13)$$

The derivation and proof is feasible but tedious. One can refer to [71] for the analogous details.

4.2. Numerical method

In this subsection, we first introduce a coordinates transformation and reformulate the rotating FGPE (1.1)-(1.2) in the new coordinates, eliminating hence the rotation term.

4.2.1. Rotating Lagrangian coordinates transformation

For any time $t \geq 0$, let $A(t)$ be the orthogonal rotational matrix defined as [16]

$$A(t) = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix}, \quad \text{if } d = 2, \quad A(t) = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{if } d = 3. \quad (4.14)$$

It is easy to check that $A^{-1}(t) = A^T(t)$ for any $t \geq 0$ and $A(0) = I$, where I is the identity matrix. For any $t \geq 0$, we introduce the *rotating Lagrangian coordinates* $\tilde{\mathbf{x}}$ as [9, 16, 38]

$$\tilde{\mathbf{x}} = A^{-1}(t)\mathbf{x} = A^T(t)\mathbf{x} \Leftrightarrow \mathbf{x} = A(t)\tilde{\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^d, \quad (4.15)$$

and we denote by $\phi := \phi(\tilde{\mathbf{x}}, t)$ the wave function in the new coordinates

$$\phi(\tilde{\mathbf{x}}, t) := \psi(\mathbf{x}, t) = \psi(A(t)\tilde{\mathbf{x}}, t), \quad \mathbf{x} \in \mathbb{R}^d, \quad t \geq 0. \quad (4.16)$$

By some simple calculations, one can easily obtain

$$\partial_t \phi(\tilde{\mathbf{x}}, t) = \partial_t \psi(\mathbf{x}, t) + \nabla \psi(\mathbf{x}, t) \cdot \left(\dot{A}(t)\tilde{\mathbf{x}} \right) = \partial_t \psi(\mathbf{x}, t) - \Omega(x\partial_y - y\partial_x)\psi(\mathbf{x}, t), \quad (4.17)$$

$$(-\nabla^2 + m^2)^s \psi(\mathbf{x}, t) = (-\nabla^2 + m^2)^s \phi(\tilde{\mathbf{x}}, t). \quad (4.18)$$

Plugging them back into (1.1)-(1.2) gives the following FNLSE in the rotating Lagrangian coordinates

$$i\partial_t \phi(\tilde{\mathbf{x}}, t) = \left[\frac{1}{2}(-\nabla^2 + m^2)^s + \mathcal{W}(\tilde{\mathbf{x}}, t) + \beta|\phi|^2 + \lambda\tilde{\Phi}(\tilde{\mathbf{x}}, t) \right] \phi(\tilde{\mathbf{x}}, t), \quad \tilde{\mathbf{x}} \in \mathbb{R}^d, \quad t > 0, \quad (4.19)$$

$$\tilde{\Phi}(\tilde{\mathbf{x}}, t) = \tilde{\mathcal{U}} * |\phi|^2, \quad \tilde{\mathbf{x}} \in \mathbb{R}^d, \quad t \geq 0. \quad (4.20)$$

Here, $\mathcal{W}(\tilde{\mathbf{x}}, t) = V(A(t)\tilde{\mathbf{x}})$ and $\tilde{\mathcal{U}}(\tilde{\mathbf{x}}, t)$ reads as

$$\tilde{\mathcal{U}}(\tilde{\mathbf{x}}, t) = \begin{cases} \frac{1}{2^{d-1}\pi|\tilde{\mathbf{x}}|^\mu}, & 0 < \mu < d-1, & \text{Coulomb,} \\ -\delta(\tilde{\mathbf{x}}) - 3\partial_{\mathbf{m}(t)\mathbf{m}(t)}\left(\frac{1}{4\pi|\tilde{\mathbf{x}}|}\right), & & \text{3D DDI,} \\ -\frac{3}{2}(\partial_{\mathbf{m}_\perp(t)\mathbf{m}_\perp(t)} - m_3^2\nabla_\perp^2)\left(\frac{1}{2\pi|\tilde{\mathbf{x}}|}\right), & & \text{2D DDI,} \end{cases} \quad (4.21)$$

with $\mathbf{m}(t) \in \mathbb{R}^3$ defined as $\mathbf{m}(t) = A^{-1}(t)\mathbf{n} = ((m_1(t), m_2(t), m_3(t))^T$ and $\mathbf{m}_\perp(t) = (m_1(t), m_2(t))^T$.

We can clearly see that the rotation term vanishes in the new coordinates (see (4.19)). Instead, the trapping potential and the dipole axis become time-dependent. The absence of the rotating term allows us to develop a simple and efficient time-splitting scheme.

4.2.2. A time-splitting pseudo-spectral method

Here we shall consider the new equation (4.19)-(4.20) which has been reformulated in rotating Lagrangian coordinates. In a practical computation, we first truncate the problem into a bounded computational domain $\mathbf{B} = [L_{\tilde{x}}, R_{\tilde{x}}] \times [L_{\tilde{y}}, R_{\tilde{y}}] \times [L_{\tilde{z}}, R_{\tilde{z}}]$ if $d = 3$, or $\mathbf{B} = [L_{\tilde{x}}, R_{\tilde{x}}] \times [L_{\tilde{y}}, R_{\tilde{y}}]$ if $d = 2$. From $t = t_n$ to $t = t_{n+1} := t_n + \Delta t$, the equation is solved in two steps. One first considers

$$i\partial_t \phi(\tilde{\mathbf{x}}, t) = \frac{1}{2}(-\nabla^2 + m^2)^s \phi(\tilde{\mathbf{x}}, t), \quad \tilde{\mathbf{x}} \in \mathbf{B}, \quad t_n \leq t \leq t_{n+1}, \quad (4.22)$$

with periodic boundary conditions on the boundary $\partial\mathbf{B}$ for a time step Δt , then solves

$$i\partial_t\phi(\tilde{\mathbf{x}}, t) = \left[\mathcal{W}(\tilde{\mathbf{x}}, t) + \beta|\phi|^2 + \lambda\tilde{\Phi}(\tilde{\mathbf{x}}, t) \right] \phi(\tilde{\mathbf{x}}, t), \quad \tilde{\mathbf{x}} \in \mathbf{B}, \quad t_n \leq t \leq t_{n+1}, \quad (4.23)$$

$$\tilde{\Phi}(\tilde{\mathbf{x}}, t) = (\tilde{\mathcal{U}} * \tilde{\rho})(\tilde{\mathbf{x}}, t), \quad \tilde{\mathbf{x}} \in \mathbf{B}, \quad t_n \leq t \leq t_{n+1}, \quad (4.24)$$

for the same time step. Here, $\tilde{\rho}(\tilde{\mathbf{x}}, t) = |\phi(\tilde{\mathbf{x}}, t)|^2$ if $\tilde{\mathbf{x}} \in \mathbf{B}$ and $\tilde{\rho}(\tilde{\mathbf{x}}, t) = 0$ otherwise. The linear subproblem (4.22) is discretized in space by the Fourier pseudo-spectral method and integrated in time exactly in the phase space. The nonlinear subproblem (4.23)-(4.24) preserves the density pointwise, i.e. $|\phi(\tilde{\mathbf{x}}, t)|^2 \equiv |\phi(\tilde{\mathbf{x}}, t = t^n)|^2 = |\phi^n(\tilde{\mathbf{x}})|^2$, and it can be integrated exactly as

$$\phi(\mathbf{x}, t) = \exp \left\{ -i \left[(t - t_n)\beta|\phi^n(\tilde{\mathbf{x}})|^2 + \lambda \varphi(\tilde{\mathbf{x}}, t) + P(\mathbf{x}, t) \right] \right\}, \quad \tilde{\mathbf{x}} \in \mathbf{B}, \quad t_n \leq t \leq t_{n+1}, \quad (4.25)$$

$$\varphi(\tilde{\mathbf{x}}, t) = \int_{\mathbb{R}^d} \tilde{\mathcal{K}}(\tilde{\mathbf{y}}, t) \rho(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}, t^n) d\tilde{\mathbf{y}}, \quad (4.26)$$

where the time-dependent kernel $\tilde{\mathcal{K}}(\tilde{\mathbf{x}}, t)$ has the form

$$\tilde{\mathcal{K}}(\tilde{\mathbf{x}}, t) = \int_{t^n}^t \tilde{\mathcal{U}}(\tilde{\mathbf{x}}, \tau) d\tau = \begin{cases} (t - t_n)/(2^{d-1}\pi|\tilde{\mathbf{x}}|^\mu), & \text{Coloumb,} \\ -\delta(\tilde{\mathbf{x}})(t - t^n) - 3\tilde{L}_3(t)(\frac{1}{4\pi|\tilde{\mathbf{x}}|}), & \text{3D DDI,} \\ -\frac{3}{2}\tilde{L}_2(t)(\frac{1}{2\pi|\tilde{\mathbf{x}}|}), & \text{2D DDI.} \end{cases} \quad (4.27)$$

Here, the differential operators $\tilde{L}_3(t) = \int_{t^n}^t \partial_{\mathbf{m}(\tau)} \mathbf{m}(\tau) d\tau$ and $\tilde{L}_2(t) = \int_{t^n}^t \partial_{\mathbf{m}_\perp(\tau)} \mathbf{m}_\perp(\tau) d\tau$ can be actually integrated analytically and have some explicit expressions. One refers to *section 4.1* in [16] for more details. The GauSum solver is then applied to evaluate the nonlocal nonlinear interaction $\varphi(\tilde{\mathbf{x}}, t)$ (4.26). In addition, we have

$$P(\tilde{\mathbf{x}}, t) = \int_{t^n}^t \mathcal{W}(\tilde{\mathbf{x}}, \tau) d\tau = \int_{t^n}^t V(A(\tau)\tilde{\mathbf{x}}) d\tau. \quad (4.28)$$

If $V(\mathbf{x})$ is chosen as the harmonic potential (1.4), then $P(\tilde{\mathbf{x}}, t)$ can be calculated analytically. For a general potential, a numerical quadrature can be used to approximate the integral (4.28).

To simplify the notations, we only present the scheme for the 2D case. Let L and M be two even positive integers. We choose $h_{\tilde{x}} = \frac{R_{\tilde{x}} - L_{\tilde{x}}}{L}$ and $h_{\tilde{y}} = \frac{R_{\tilde{y}} - L_{\tilde{y}}}{M}$ as the spatial mesh sizes in the \tilde{x} - and \tilde{y} -directions, respectively. We define the indices and grid points sets as

$$\begin{aligned} \mathcal{T}_{LM} &= \{(\ell, m) \in \mathbb{N}^2 \mid 0 \leq \ell \leq L, 0 \leq m \leq M\}, \\ \tilde{\mathcal{T}}_{LM} &= \{(p, q) \in \mathbb{N}^2 \mid -L/2 \leq p \leq L/2 - 1, -M/2 \leq q \leq M/2 - 1\}, \\ \mathcal{G}_{\tilde{x}\tilde{y}} &= \{(\tilde{x}_\ell, \tilde{y}_m) =: (L_x + \ell h_x, L_y + m h_y), (\ell, m) \in \mathcal{T}_{LM}\}. \end{aligned}$$

We introduce the following functions

$$W_{pq}(\tilde{x}, \tilde{y}) = e^{i\mu_p^{\tilde{x}}(\tilde{x} - L_{\tilde{x}})} e^{i\mu_q^{\tilde{y}}(\tilde{y} - L_{\tilde{y}})}, \quad (p, q) \in \tilde{\mathcal{T}}_{LM},$$

with

$$\mu_p^{\tilde{x}} = \frac{2\pi p}{R_{\tilde{x}} - L_{\tilde{x}}}, \quad \mu_q^{\tilde{y}} = \frac{2\pi q}{R_{\tilde{y}} - L_{\tilde{y}}}, \quad (p, q) \in \tilde{\mathcal{T}}_{LM}.$$

Let $f_{\ell m}^n$ ($f = \phi, \varphi$ or P) be the approximation of $f(\tilde{x}_\ell, \tilde{y}_m, t_n)$ for $(\ell, m) \in \mathcal{T}_{LM}$ and $n \geq 0$. We denote by ϕ^n the solution at time $t = t_n$, with components $\{\phi_{\ell m}^n, (\ell, m) \in \mathcal{T}_{LM}\}$. We take the initial data as $\phi_{\ell m}^0 = \phi_0(\tilde{x}_\ell, \tilde{y}_m)$, for $(\ell, m) \in \mathcal{T}_{LM}$. A second-order time-splitting Fourier pseudo-spectral (TSFP) method

to solve (4.19)-(4.20) is given by

$$\phi_{\ell m}^{(1)} = \sum_{p=-L/2}^{L/2-1} \sum_{q=-M/2}^{M/2-1} e^{-\frac{i\Delta t}{4}[(\mu_{\tilde{p}}^x)^2 + (\mu_{\tilde{q}}^y)^2 + m^2]^s} \widehat{(\phi^n)}_{pq} W_{pq}(\tilde{x}_\ell, \tilde{y}_m), \quad (4.29)$$

$$\phi_{\ell m}^{(2)} = \phi_{\ell m}^{(1)} \exp \left\{ -i \left[\Delta t \beta |\phi_{\ell m}^n|^2 + \lambda \varphi_{\ell m}^{n+1} + P_{\ell m}^{n+1} \right] \right\}, \quad (4.30)$$

$$\phi_{\ell m}^{n+1} = \sum_{p=-L/2}^{L/2-1} \sum_{q=-M/2}^{M/2-1} e^{-\frac{i\Delta t}{4}[(\mu_{\tilde{p}}^x)^2 + (\mu_{\tilde{q}}^y)^2 + m^2]^s} \widehat{(\phi^{(2)})}_{pq} W_{pq}(\tilde{x}_\ell, \tilde{y}_m). \quad (4.31)$$

Here, $\widehat{(\phi^n)}_{pq}$ and $\widehat{(\phi^{(2)})}_{pq}$ are the discrete Fourier series coefficients of the vectors ϕ^n and $\phi^{(2)}$, respectively. This method is referred to as *TS2-GauSum*. The *TS2-GauSum* method (4.29)-(4.31) is explicit, efficient, simple to implement, unconditionally stable and can be easily extended to high-order time-splitting schemes.

4.3. Numerical results

In this subsection, we present some numerical results for the dynamics of the FNLSE/FGPE solved by *TS2-GauSum*. To this end, unless stated, we let $m = 0$, $\Omega = 0$, $d = 2$ and choose the computational domain as $\mathbf{B} = [-16, 16] \times [-16, 16]$. The mesh sizes in space and time are chosen as $h_x = h_y = \frac{1}{8}$ and $\Delta t = 10^{-3}$, respectively. The trapping potential $V(\mathbf{x})$ is chosen as (1.4) with $\gamma_x = \gamma_y = 1$. The nonlocal interaction is of Coulomb-type with $\mu = 1$. The initial data is set to

$$\psi_0(\mathbf{x}) = \phi_g^s(\mathbf{x} - \mathbf{x}_0) e^{i v_0(0.8x + 0.5y)}, \quad (4.32)$$

where ϕ_g^s is the ground state of the FNLSE with the fractional order s . Starting from the ground state $\phi_g^s(\mathbf{x})$, we shift it by $\mathbf{x}_0 \in \mathbb{R}^2$ and/or imprint an initial momentum as shown above.

Example 4.1. Dynamics of the FNLSE ($\mathbf{x}_0 = (0, 0)^T$). In this example, let $\beta = 0$, $\lambda = -1$, $v_0 = 1$ and $\mathbf{x}_0 = (0, 0)^T$ in (4.32). We study two cases in (4.32): **Case I:** $s = 1$, **Case II:** $s \neq 1$.

Figure 5 and 6 show the dynamics of mass, energy, centre of mass, condensate widths of the FNLSE with different fractional orders s . We can observe that (i) The mass and total energy are well conserved. (ii) The fractional order significantly affects the dynamics of the FNLSE. As we know, for the classical NLSE ($s = 1$), the density profile retains its initial shape, meanwhile swings periodically in the harmonic trap (cf. Fig. 5 (a)). However, for the fractional case ($s \neq 1$), the density profile is quite different from the initial profile. For the subdispersion case, $s < 1$, the decoherence emerges, i.e. the loss of solitary profile, and it becomes stronger when $|s - 1|$ is larger. For superdispersion, i.e. $s > 1$, there is much less decoherence observed. The density profile would exhibit damped oscillations around what appears to be a rescaled ground state, which behaves similarly as the breather solutions of the classical NLSE. (iii) For both cases, the decoherence is weak and turbulence (the high frequencies) does not emerge, letting alone the chaotic dynamics. The turbulence and/or chaotic dynamics might emerge if the initially imprinted momentum is large enough.

Example 4.2. Dynamics of the FNLSE with position shifts in initial data. With fixed $s = 0.75$ (subdispersion) and $v_0 = 0$ in (4.32), we study the following four cases:

- **Case I.** Linear fractional Schrödinger equation. Let $\beta = \lambda = 0$, $\mathbf{x}_0 = (1, 1)^T$.
- **Case II.** Linear fractional Schrödinger equation. Let $\beta = \lambda = 0$, $\mathbf{x}_0 = (3, 3)^T$.
- **Case III.** FNLSE with purely short-range interaction. Let $\beta = 50$, $\lambda = 0$, $\mathbf{x}_0 = (3, 3)^T$.
- **Case IV.** FNLSE with purely long-range interaction. Let $\beta = 0$, $\lambda = 10$, $\mathbf{x}_0 = (3, 3)^T$.

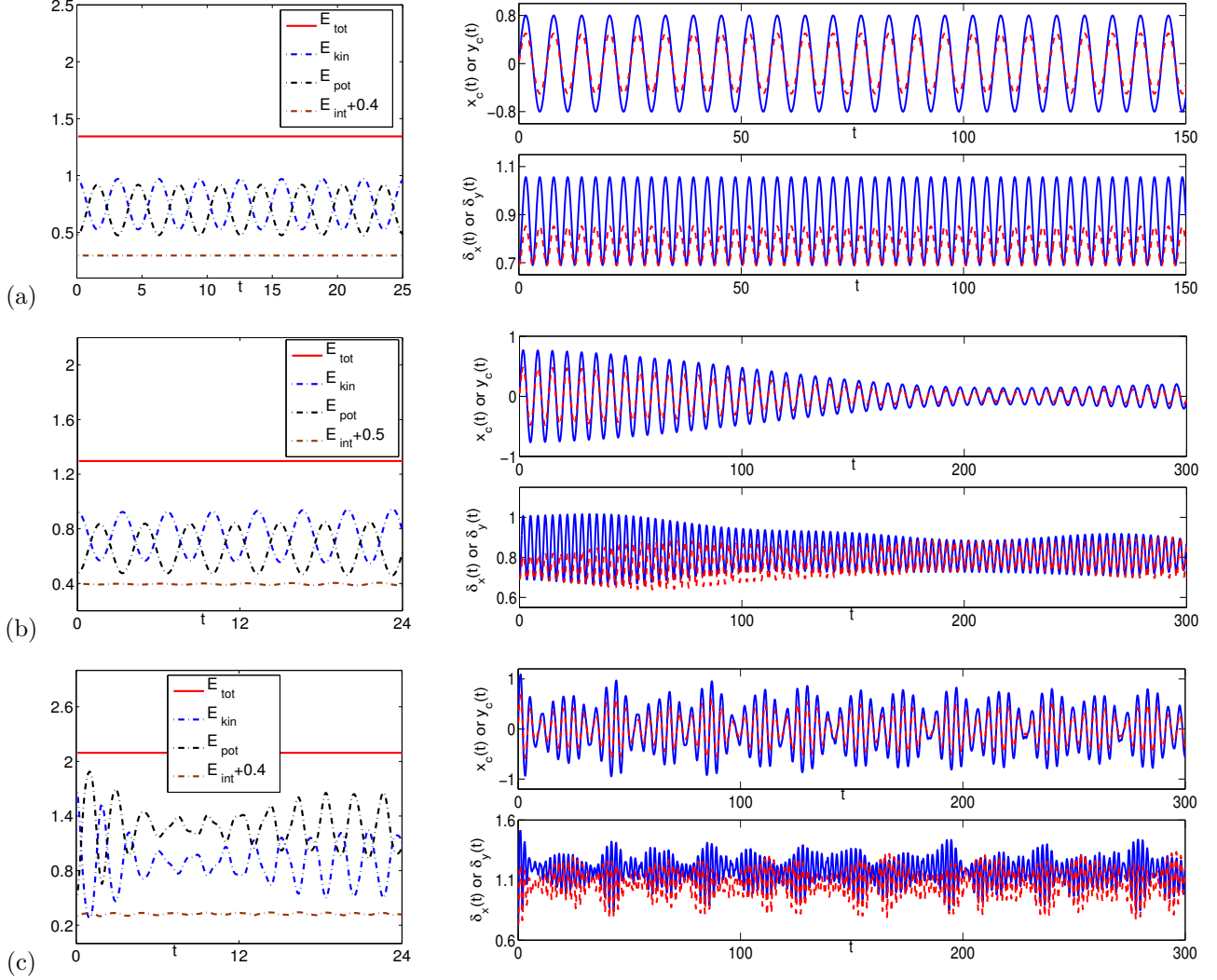


Figure 5: Time evolution of the energies, centre of mass and condensate width in example 4.1 for case I for: (a) $s = 1$ (standard case), (b) $s = 0.95$ (subdispersion case) and (c) $s = 1.5$ (superdispersion case).

Figure 7 shows the dynamics of the mass, energy, centre of mass, condensate widths, while Figure 8 shows the contour plot of the density $|\psi(\mathbf{x}, t)|^2$ at different times. Similarly to Example 4.1, we can see that (i) For the FNLSE, the density profile no longer retains its initial shape as in the classical NLSE. The density profile also oscillates around the center of the trap and decoherence emerges. (ii) The dynamics of the wave function depends crucially on the initial shift \mathbf{x}_0 . If the initial shift is small, the initial shape is changed slightly, i.e. the decoherence is small (cf. Fig. 8 (a)), while for large shifts, the decoherence appears very quickly. Turbulence and chaotic dynamics might also occur for a large \mathbf{x}_0 in the linear FSE (cf. Fig. 8 (b)). (iii) Both the short- and long-range nonlinear interactions can reduce and/or delay the emergence of decoherence and suppress the wave function from chaotic dynamics. Turbulence emerges in the FNLSE with pure local nonlinearity (see Fig. 8 (c)), while the decoherence is weaker in the FNLSE with pure nonlocal nonlinearity. The density profile would actually oscillate like a breather (cf. Fig. 8 (d)). It would also be interesting to investigate the decoherence and turbulence properties in the superdispersion case $s > 1$ and analyze how they are affected through a rotation effect. This will be analyzed in future research. Our results are in accordance with those showed in [51].

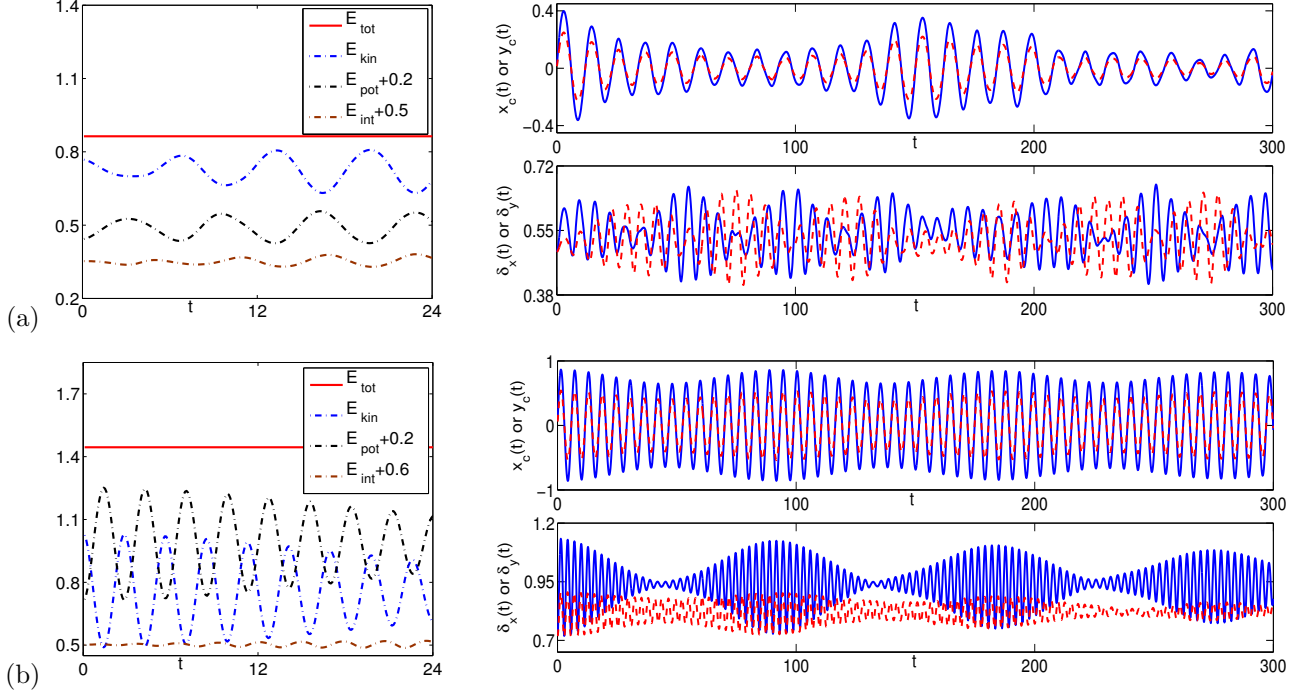


Figure 6: Time evolution of the energies, centre of mass and condensate width in example 4.1 for case II for: (a) $s = 0.5$ (subdispersion case), (b) $s = 1.1$ (superdispersion case).

5. Conclusion

In this paper, we proposed efficient and robust numerical methods for computing the ground states and dynamics of the FNLSE equation with an angular momentum and nonlocal interaction potentials. Existence and non-existence of the ground states were presented and dynamical laws for the mass, energy, angular momentum and center of mass were obtained.

We then studied the ground states and dynamics of the FNLSE numerically. It was found that the fractional order s affects both the ground states and dynamics in a significant way. The ground states become more peaked as $s < 1$ tends smaller, corresponding here to subdispersion. For the superdispersion case, i.e. $s > 1$, the creation of a giant vortex can be observed for a fast rotating system, which is totally different from the behavior of the classical GPE. Critical values of the rotating frequencies to create the first vortex solution are numerically found to depend on s . For the dynamics, decoherence as well as turbulence were observed in the FNLSE when an initial data is prepared from a ground state with imprinted phase shift and/or position shift. It is shown that the smaller the fractional exponent s is, the easier the decoherence emerges. The larger the initial shift is, the easier the turbulence and chaotic dynamics arise. Furthermore, the presence of repulsive nonlinearities, both local and nonlocal, can suppress the “peaking” effects of the ground states and the decoherence/turbulence observed in the dynamics.

It is worthwhile to remark that the ground states of the FNLSE decay only algebraically as $|\mathbf{x}| \rightarrow \infty$ when the external potential $V(\mathbf{x})$ is bounded [36] and $\beta < 0$. A very large computational domain is necessary for both the ground state computation and the dynamics [52]. It would be interesting and crucial to derive a fractional version of the free boundary conditions such as the transient BC, absorbing BC and also the PML [3] for the FNLSE.

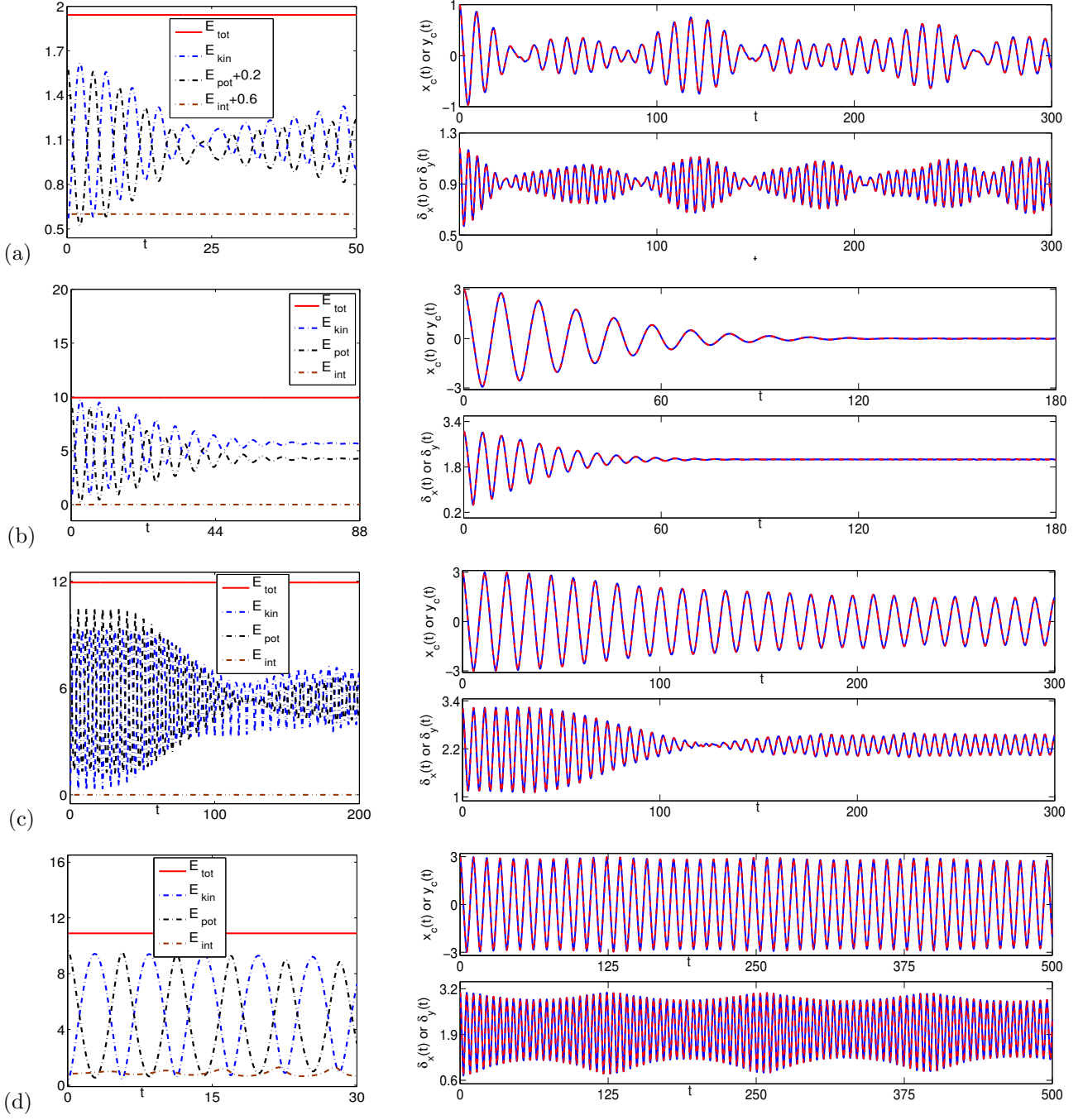


Figure 7: Time evolution of the energies, centre of mass and condensate width in example 4.2 for cases I to IV (from top to bottom). Here, we consider a subdispersion case for $s = 0.75$.

Finally, let us emphasize that the time and space fractional NLSE, for $0 < \gamma < 1$,

$$i\partial_t^\gamma \psi(\mathbf{x}, t) = \left[\frac{1}{2} (-\nabla^2 + m^2)^s + V(\mathbf{x}) + \beta |\psi(\mathbf{x}, t)|^2 + \lambda \Phi(\mathbf{x}, t) - \Omega L_z \right] \psi(\mathbf{x}, t), \quad (5.33)$$

$$\Phi(\mathbf{x}, t) = \mathcal{U} * |\psi(\mathbf{x}, t)|^2, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \quad d \geq 2. \quad (5.34)$$

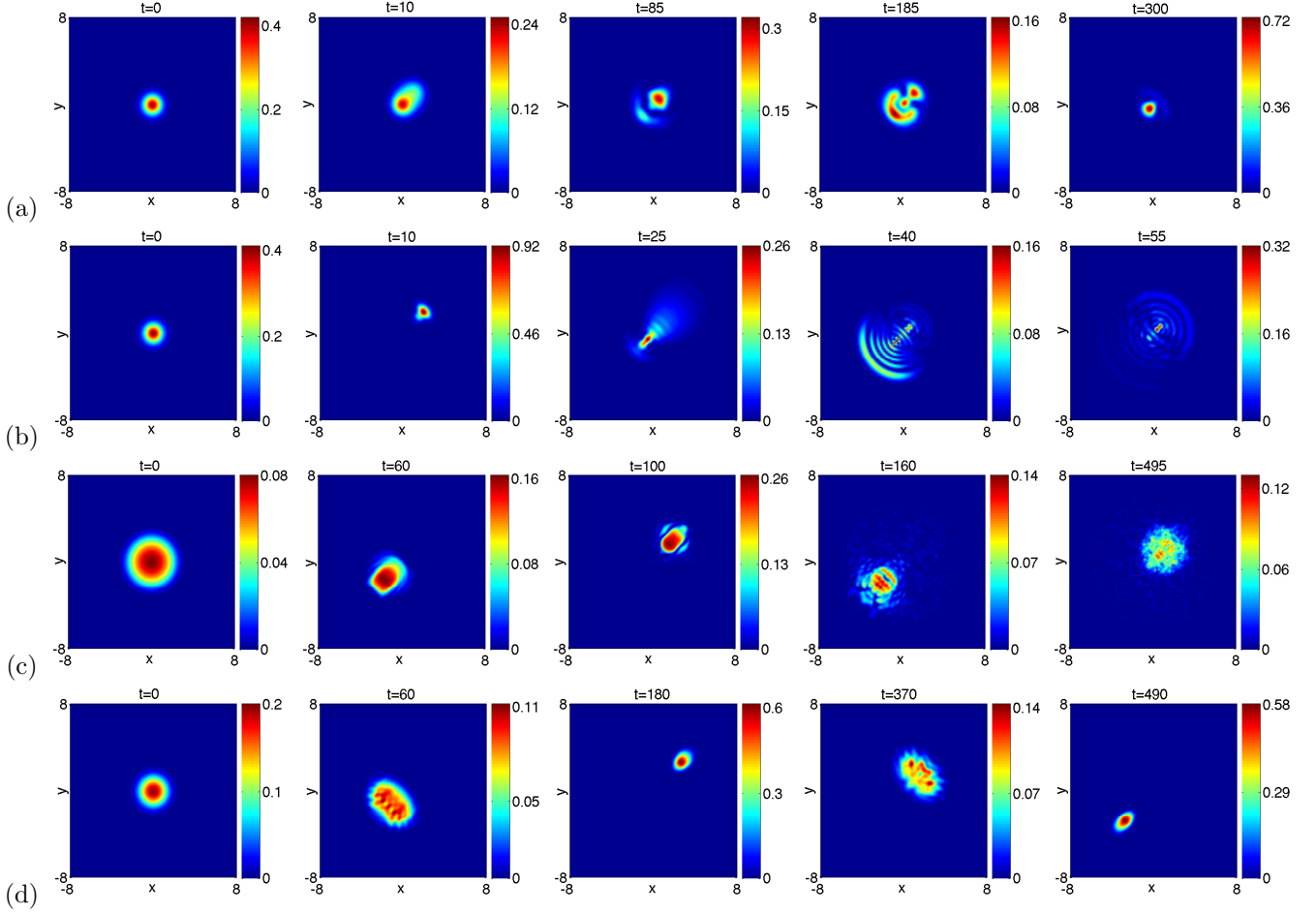


Figure 8: Contour plots of the density $|\psi(\mathbf{x}, t)|^2$ at different times in example 4.2 for cases I to IV (from top to bottom). Here, we consider $s = 0.75$ (subdispersion).

is also very interesting for some applications [31, 39, 53, 54, 63, 78]. The next step of our work would consist in analyzing efficient and accurate numerical methods for solving FNLSEs both in space and time and understand their behavior and properties.

Acknowledgements

We acknowledge the support from the ANR project BECASIM ANR-12-MONU-0007-02 (X. Antoine and Q. Tang), the ANR-FWF Project Lodiquas ANR-11-IS01-0003, the ANR project Moonrise ANR-14-CE23-0007-01 and the Natural Science Foundation of China grants 11261065, 91430103 and 11471050 (Y. Zhang). We are also grateful to Prof. Weizhu Bao and Dr. Yongyong Cai for valuable suggestions.

Appendix A. Proof of lemma 4.1

Let us introduce $\mathbf{k} = (k_x, k_y, k_z)^T$ if $d = 3$, and $\mathbf{k} = (k_x, k_y)^T$ if $d = 2$. Let $J_z = y\partial_x - x\partial_y$ and $\widehat{J}_z = k_y\partial_{k_x} - k_x\partial_{k_y}$. Then, we have $\widehat{J}_z\psi = \widehat{J}_z\psi$. Differentiating (4.2), noticing (1.1) and using the

Plancherel's formula, we have

$$\begin{aligned}\frac{d}{dt}\langle L_z \rangle(t) &= \langle L_z \psi_t, \psi \rangle + \langle L_z \psi, \psi_t \rangle = -\langle i\psi_t, J_z \psi \rangle - \langle J_z \psi, i\psi_t \rangle \\ &= \frac{1}{(2\pi)^d} \left\{ -\langle \frac{1}{2}(|\mathbf{k}|^2 + m^2)^s \widehat{\psi} + \widehat{\mathcal{V}}\psi, \widehat{J}_z \widehat{\psi} \rangle - \langle \widehat{J}_z \widehat{\psi}, \frac{1}{2}(|\mathbf{k}|^2 + m^2)^s \widehat{\psi} + \widehat{\mathcal{V}}\psi \rangle \right\},\end{aligned}\quad (\text{A.1})$$

with $\mathcal{V}\psi := V\psi + \lambda\Phi\psi$. The rotation and local nonlinear terms cancel. We omit both for brevity.

By integrating the above equation by parts, we have

$$\begin{aligned}\frac{d}{dt}\langle L_z \rangle(t) &= \frac{1}{(2\pi)^d} \left\{ -\langle \frac{1}{2}(|\mathbf{k}|^2 + m^2)^s \widehat{\psi} + \widehat{\mathcal{V}}\psi, \widehat{J}_z \widehat{\psi} \rangle + \langle \widehat{\psi}, \widehat{J}_z \left(\frac{1}{2}(|\mathbf{k}|^2 + m^2)^s \widehat{\psi} + \widehat{\mathcal{V}}\psi \right) \rangle \right\} \\ &= \frac{1}{(2\pi)^d} \left\{ \langle \widehat{\psi}, \widehat{J}_z(\widehat{\mathcal{V}}\psi) \rangle - \langle \widehat{\mathcal{V}}\psi, \widehat{J}_z \widehat{\psi} \rangle \right\} = \frac{1}{(2\pi)^d} \left\{ -\langle \widehat{J}_z \widehat{\psi}, \widehat{\mathcal{V}}\psi \rangle - \langle \widehat{\mathcal{V}}\psi, \widehat{J}_z \widehat{\psi} \rangle \right\} \\ &= -\langle J_z \psi, \mathcal{V}\psi \rangle - \langle \mathcal{V}\psi, J_z \psi \rangle = \langle |\psi|^2, J_z \mathcal{V} \rangle = \int_{\mathbb{R}^d} |\psi|^2 (y\partial_x - x\partial_y) (V(\mathbf{x}) + \lambda\Phi(\mathbf{x}, t)) d\mathbf{x}.\end{aligned}\quad (\text{A.2})$$

Therefore, by adapting the polar/cylindrical coordinates transformation in 2D/3D and noticing $y\partial_x - x\partial_y = -\partial_\theta$, one can obtain

$$I_1 =: \int_{\mathbb{R}^d} |\psi|^2 (y\partial_x - x\partial_y) V(\mathbf{x}) d\mathbf{x} = 0, \quad (\text{A.3})$$

provide that $V(\mathbf{x})$ is radially/cylindrically symmetric in 2D/3D. Now that

$$I_2 =: \int_{\mathbb{R}^d} |\psi|^2 (y\partial_x - x\partial_y) \Phi(\mathbf{x}, t) d\mathbf{x} = \frac{1}{(2\pi)^d} \langle |\widehat{\psi}|^2, \widehat{J}_z \widehat{\Phi} \rangle = \int_{\mathbb{R}^d} \widehat{\mathcal{U}}(\mathbf{k}) |\widehat{\psi}|^2 (k_y \partial_{k_x} - k_x \partial_{k_y}) |\widehat{\psi}|^2 d\mathbf{k}, \quad (\text{A.4})$$

applying the polar/cylindrical coordinates transformation in 2D/3D in the Fourier space, it is easily to get $I_2 = 0$ if $\widehat{\mathcal{U}}(\mathbf{k})$ in (1.5) is chosen as the Coulomb-type interaction or DDI with $\mathbf{n} = (0, 0, 1)^T$. \square

Appendix B. Proof of lemma 4.2

Step 1: By differentiating (4.5) and noticing (1.1), we have

$$\begin{aligned}\dot{\mathbf{x}}_c(t) &= \frac{d}{dt} \langle \mathbf{x}\psi, \psi \rangle = \frac{1}{i} \langle \mathbf{x} i\psi_t, \psi \rangle + i \langle \mathbf{x}\psi, i\psi_t \rangle = i [\langle \mathbf{x}\psi, i\psi_t \rangle - \langle \mathbf{x} i\psi_t, \psi \rangle] \\ &= \frac{i}{2} [\langle \mathbf{x}\psi, (-\Delta + m^2)^s \psi \rangle - \langle \mathbf{x} (-\Delta + m^2)^s \psi, \psi \rangle] - \Omega [\langle \mathbf{x} J_z \psi, \psi \rangle + \langle \psi, \mathbf{x} J_z \psi \rangle].\end{aligned}\quad (\text{B.1})$$

An integration by parts and an application of Plancherel's formula lead to

$$\begin{aligned}\dot{\mathbf{x}}_c(t) &= \frac{i}{2} \frac{1}{(2\pi)^d} \left\{ \langle i\nabla_{\mathbf{k}} \widehat{\psi}, (|\mathbf{k}|^2 + m^2)^s \widehat{\psi} \rangle - \langle i\nabla_{\mathbf{k}} [(|\mathbf{k}|^2 + m^2)^s \widehat{\psi}(\mathbf{k})], \widehat{\psi} \rangle \right\} + \Omega \langle \psi, J_z \mathbf{x}\psi \rangle \\ &= \frac{s}{(2\pi)^d} \langle (|\mathbf{k}|^2 + m^2)^{s-1} \mathbf{k} \widehat{\psi}, \widehat{\psi} \rangle + \Omega J \mathbf{x}_c.\end{aligned}\quad (\text{B.2})$$

Note that (B.2) is well-defined for $\forall s > 0$. If $s = 1$, (B.2) yields

$$\dot{\mathbf{x}}_c(t) - \Omega J \mathbf{x}_c = \frac{1}{(2\pi)^d} \langle \mathbf{k} \widehat{\psi}, \widehat{\psi} \rangle = i \langle \psi, \nabla \psi \rangle = i \langle G * \psi, \nabla \psi \rangle, \quad (\text{B.3})$$

with $G(\mathbf{x}) = \delta(\mathbf{x})$. If $0 < s < 1$, we have

$$(|\mathbf{k}|^2 + m^2)^{s-1} = \frac{1}{c_s} \int_0^\infty \lambda^{-s} e^{-\pi(|\mathbf{k}|^2 + m^2)\lambda} d\lambda, \quad \text{with} \quad c_s = \Gamma(1-s)/\pi^{1-s}. \quad (\text{B.4})$$

Hence, one gets

$$\begin{aligned}
s \mathcal{F}^{-1} \left((|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\psi} \right) &= s \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\psi} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \\
&= \frac{s}{c_s} \int_0^\infty \lambda^{-s} e^{-\pi \lambda m^2} \left[\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\psi} e^{-\pi \lambda |\mathbf{k}|^2} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \right] d\lambda = \frac{s}{c_s} \int_0^\infty \lambda^{-s} e^{-\pi \lambda m^2} \left[\psi * \mathcal{F}^{-1} \left(e^{-\pi \lambda |\mathbf{k}|^2} \right) \right] d\lambda \\
&= \frac{s}{c_s (2\pi)^d} \int_0^\infty \lambda^{-s} e^{-\pi \lambda m^2} \int_{\mathbb{R}^d} \lambda^{-\frac{d}{2}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\pi\lambda}} \psi(\mathbf{y}) d\mathbf{y} d\lambda \\
&= \frac{s}{c_s (2\pi)^d} \int_{\mathbb{R}^d} \psi(\mathbf{y}) \left[\int_0^\infty \lambda^{-\frac{2s+d}{2}} e^{-\pi \lambda m^2} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\pi\lambda}} d\lambda \right] d\mathbf{y} =: (G * \psi)(\mathbf{x}), \tag{B.5}
\end{aligned}$$

with

$$G(\mathbf{x}) = \frac{s}{c_s (2\pi)^d} \int_0^\infty \lambda^{-\frac{2s+d}{2}} e^{-\pi \lambda m^2} e^{-\frac{|\mathbf{x}|^2}{4\pi\lambda}} d\lambda = \frac{2^{s-d/2} s}{\Gamma(1-s) \pi^{d/2}} \left(\frac{m}{|\mathbf{x}|} \right)^{\frac{d}{2}+s-1} K_{\frac{d}{2}+s-1}(m|\mathbf{x}|), \tag{B.6}$$

where $K_v(z)$ is the modified Bessel function of the second-kind and order v defined by (4.10). Finally, we obtain

$$\dot{\mathbf{x}}_c(t) - \Omega J \dot{\mathbf{x}}_c = \frac{s}{(2\pi)^d} \langle (|\mathbf{k}|^2 + m^2)^{s-1} \mathbf{k} \widehat{\psi}, \widehat{\psi} \rangle = \left\langle s \mathcal{F}^{-1}(|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\psi}, \mathcal{F}^{-1}(\mathbf{k} \widehat{\psi}) \right\rangle = i \langle (G * \psi), \nabla \psi \rangle. \tag{B.7}$$

Step 2: Let us consider the second-order derivative of (B.2). By (B.5), we have

$$\begin{aligned}
\ddot{\mathbf{x}}_c(t) - \Omega J \ddot{\mathbf{x}}_c &= 2s \operatorname{Re} \left(C_d \langle \mathbf{k} (|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\psi}_t, \widehat{\psi} \rangle \right) = 2s \operatorname{Im} \left(C_d \langle i \widehat{\psi}_t, \mathbf{k} (|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\psi} \rangle \right) \\
&= 2 \operatorname{Im} \left(C_d \langle s \mathcal{F}^{-1} \left((|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\mathcal{V}} \widehat{\psi} \right), \mathcal{F}^{-1}(\mathbf{k} \widehat{\psi}) \rangle \right) + s \Omega \left[C_d \langle \widehat{\psi}, \mathbf{k} (|\mathbf{k}|^2 + m^2)^{s-1} \widehat{J}_z \widehat{\psi} \rangle \right. \\
&\quad \left. + C_d \langle \widehat{\psi}, \widehat{\psi} \widehat{J}_z (\mathbf{k} (|\mathbf{k}|^2 + m^2)^{s-1}) \rangle - C_d \langle \mathbf{k} (|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\psi}, \widehat{J}_z \widehat{\psi} \rangle \right] \\
&= 2 \operatorname{Re} \left(\langle G * (\mathcal{V} \psi), \nabla \psi \rangle \right) + s \Omega C_d \langle (|\mathbf{k}|^2 + m^2)^{s-1} \widehat{\psi}, \widehat{\psi} \widehat{J}_z \mathbf{k} \rangle \\
&= 2 \operatorname{Re} \left(\langle G * (\mathcal{V} \psi), \nabla \psi \rangle \right) + \Omega J (\dot{\mathbf{x}}_c - \Omega J \dot{\mathbf{x}}_c).
\end{aligned}$$

Hence, we obtain

$$\ddot{\mathbf{x}}_c(t) - 2\Omega J \dot{\mathbf{x}}_c + \Omega^2 J^2 \dot{\mathbf{x}}_c = 2 \operatorname{Re} \left(\langle G * (\mathcal{V} \psi), \nabla \psi \rangle \right), \tag{B.8}$$

ending hence the proof. \square

References

- [1] G. L. AKI, P. A. MARKOWICH AND C. SPARBER, *Classical limit for semi-relativistic Hartree system*, J. Math. Phys., 49 (2008), 102–110.
- [2] P. AMORE, F. M. FERNÁNDEZ, C. P. HOFMANN AND R. A. SÁENZ, *Collocation method for fractional quantum mechanics*, J. Math. Phys., 51 (2010), 122101.
- [3] X. ANTOINE, A. ARNOLD, C. BESSE, M. EHRHARDT AND A. SCHÄDLE, *A review of transparent and artificial boundary conditions techniques for linear and nonlinear Schrödinger equations*, Commun. Comput. Phys., 4 (2008), 729–796.
- [4] X. ANTOINE, W. BAO AND C. BESSE, *Computational methods for the dynamics of the nonlinear Schrödinger/Gross-Pitaevskii equations*, Comput. Phys. Commun., 184 (2013), 2621–2633.
- [5] X. ANTOINE AND R. DUBOSCQ, *Robust and efficient preconditioned Krylov spectral solvers for computing the ground states of fast rotating and strongly interacting Bose-Einstein condensates*, J. Comput. Phys., 258 (2014), 509–523.
- [6] X. ANTOINE AND R. DUBOSCQ, *GPELab, a Matlab toolbox to solve Gross-Pitaevskii equations I: computation of stationary solutions*, Comput. Phys. Commun., 185 (2014), 2969–2991.
- [7] X. ANTOINE AND R. DUBOSCQ, *GPELab, a Matlab toolbox to solve Gross-Pitaevskii equations II: dynamics and stochastic simulations*, Comput. Phys. Commun., 193 (2015), 95–117.

- [8] X. ANTOINE AND R. DUBOSCQ, *Modeling and computation of Bose-Einstein condensates: stationary states, nucleation, dynamics, stochasticity*, in Nonlinear Optical and Atomic Systems: at the Interface of Mathematics and Physics, Lecture Notes in Mathematics, 2146, pp. 49-145, Springer.
- [9] P. ANTONELLI, D. MARAHRENS AND C. SPARBER, *On the Cauchy problem for nonlinear Schrödinger equations with rotation*, Disc. Contin. Dyn. Syst. A, 32 (2012), 703–715.
- [10] W. BAO, N. B. ABDALLAH, Y. CAI, *Gross-Pitaevskii Poisson equations for dipolar Bose-Einstein condensate with anisotropic confinement*, SIAM J. Math. Anal., 44 (2012), 1713–1741.
- [11] W. BAO, Y. CAI, *Mathematical theory and numerical methods for Bose-Einstein condensation*, Kinet. Relat. Mod., 6 (2013), 1–135.
- [12] W. BAO, Y. CAI, H. WANG, *Efficient numerical methods for computing ground states and dynamics of dipolar Bose-Einstein condensates*, J. Comput. Phys., 229 (2010), 7874–7892.
- [13] W. BAO AND X. DONG, *Numerical methods for computing ground state and dynamics of nonlinear relativistic Hartree equation for boson stars*, J. Comput. Phys., 230 (2011), 5449–5469.
- [14] W. BAO, H. JIAN, N. J. MAUSER AND Y. ZHANG, *Dimension reduction of the Schrödinger equation with Coulomb and anisotropic confining potentials*, SIAM J. Appl. Math., 73 (2013), 2100–2123.
- [15] W. BAO, S. JIANG, Q. TANG AND Y. ZHANG, *Computing the ground state and dynamics of the nonlinear Schrödinger equation with nonlocal interactions via the nonuniform FFT*, J. Comput. Phys., 296 (2015), 72–89.
- [16] W. BAO, D. MARAHRENS, Q. TANG AND Y. ZHANG, *A simple and efficient numerical method for computing the dynamics of rotating Bose-Einstein condensates via rotating Lagrangian coordinates*, SIAM J. Sci. Comput., 35 (2013), A2671–A2695.
- [17] W. BAO, Q. TANG AND Y. ZHANG, *Accurate and efficient numerical methods for computing ground states and dynamics of dipolar Bose-Einstein condensates via the nonuniform FFT*, arXiv:1504.02897.
- [18] S. S. BAYIN, *On the consistency of solutions of the space fractional Schrödinger equation*, J. Math. Phys., 53 (2012), 042105.
- [19] S. S. BAYIN, *Comment on “On the consistency of solutions of the space fractional Schrödinger equation”*, J. Math. Phys., 54 (2013), 074101.
- [20] C. BESSE, *A relaxation scheme for nonlinear Schrödinger equation*, SIAM J. Numer. Anal., 42 (2004), 934–952.
- [21] C. BESSE, G. DUJARDIN AND I. LACROIX-VIOLET, *High-order exponential integrators for nonlinear Schrödinger equations with application to rotating Bose-Einstein condensates*, 2015. (hal-01170888).
- [22] J. P. BOUCHAUD AND A. GEORGES, *Anomalous diffusion in disordered media: statistical mechanics, models and applications*, Phys. Rep., 195 (1990), 127–193.
- [23] R. CARLES, P. A. MARKOWICH AND C. SPARBER, *On the Gross-Pitaevskii equation for trapped dipolar quantum gases*, Nonlinearity, 21 (2008), 2569–2590.
- [24] Y. CHO, H. HAJAIEJ, G. HWANG AND T. OZAWA, *On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity*, Funkcialaj Ekvacioj, 56 (2013), 193–224.
- [25] Y. CHO, H. HAJAIEJ, G. HWANG AND T. OZAWA, *On the orbital stability of fractional Schrödinger equations*, Commun. Pure Appl. Anal., 13 (2014), 1267–1282.
- [26] Y. CHO AND T. OZAWA, *On the semi-relativistic Hartree-type equation*, SIAM J. Math. Anal., 38 (2006), 1060–1074.
- [27] V. COTI ZELATI AND M. NOLASCO, *Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations*, Rend. Lincei Mat. Appl. 22 (2011), 51–72.
- [28] I. DANAILA, P. KAZEMI, *A new Sobolev gradient method for direct minimization of the Gross-Pitaevskii energy with rotation*, SIAM J. Sci. Comput., 32 (2010), 2447–2467.
- [29] Q. DU, M. GUNZBURGER, R. B. LEHOUCQ AND K. ZHOU, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Rev., 54 (2012), 667–696.
- [30] A. ELGART AND B. SCHLEIN, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math., 60 (2007), 500–545.
- [31] H. ERTIK, D. DEMIRHAN, H. SIRIN AND F. BUYUKKILIC, *Time fractional development of quantum systems*, J. Math. Phys., 51 (2010), 082102.
- [32] H. ERTIK, H. SIRIN, D. DEMIRHAN AND F. BUYUKKILIC, *Fractional mathematical investigation of Bose-Einstein condensation in dilute ^{87}Rb , ^{23}Na and ^7Li atomic gases*, Int. J. Mod. Phys. B, 26 (2012), 1250096.
- [33] L. EXL, N. J. MAUSER AND Y. ZHANG, *Accurate and efficient computation of nonlocal potentials based on Gaussian-sum approximation*, arXiv:1501.04438.
- [34] B. FENG, *Ground state for the fractional Schrödinger equation*, Electron. J. Differ. Equ., 127 (2013), 1–11.
- [35] R. P. FEYNMAN AND A. R. HIBBS, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.
- [36] R. L. FRANK AND E. LENZMANN, *Uniqueness of nonlinear ground states for fractional Laplacians in \mathbb{R}* , Acta Math., 210 (2013), 261–318.
- [37] J. FRÖHLICH AND E. LENZMANN, *Blowup for nonlinear wave equations describing boson stars*, Commun. Pure Appl. Math., 60 (2007), 16911705.
- [38] J. J. GARCÍA-RIPOLL, V. M. PÉREZ-GARCÍA AND V. VEKSLERCHIK, *Construction of exact solution by spatial translations in inhomogeneous nonlinear Schrödinger equations*, Phys. Rev. E, 64 (2001), 056602.
- [39] R. GARRAPPA, I. MORET AND M. POPOLIZIO, *Solving the time fractional Schrödinger equation by Krylov projection methods*, J. Comput. Phys., 293 (2015), 115–134.
- [40] B. GUO AND Z. HUO, *Global well-posedness for the fractional nonlinear Schrödinger equation*, Comm. Partial Diff. Equ., 36 (2011), 247–255.
- [41] B. GUO, X. PU AND F. HUANG, *Fractional Partial Differential Equations and their Numerical Solutions*, World Scientific, Singapore, 2015.
- [42] X. GUO AND M. XU, *Some physical applications of fractional Schrödinger equation*, J. Math. Phys., 47 (2006), 082104.

- [43] E. HAWKINS AND J. M. SCHWARZ, *Comment on “On the consistency of solutions of the space fractional Schrödinger equation”*, J. Math. Phys., 54 (2013), 014101.
- [44] B. I. HENRY, T. A. M. LANGLANDS AND P. STRAKA, *An Introduction to Fractional Diffusion*, in: R. L. Dewar and F. Detering(ed.) *Complex Physical, Biophysical and Econophysical Systems*, World Scientific Lecture Notes in Complex Systems, Vol. 9, World Scientific, Hackensack, NJ, 2010.
- [45] Y. HONG AND Y. SIRE, *On fractional Schrödinger equation in Sobolev spaces*, arXiv: 1501.01414.
- [46] A. D. IONESCU AND F. PUSATERI, *Nonlinear fractional Schrödinger equations in one dimension*, J. Funct. Anal., 266 (2014), 139–176.
- [47] M. JENG, S.-L.-Y. XU, E. HAWKINS AND J. M. SCHWARZ, *On the nonlocality of the fractional Schrödinger equation*, J. Math. Phys., 51 (2010), 062102.
- [48] S. JIANG, L. GREENGARD AND W. BAO, *Fast and accurate evaluation of dipolar interaction in Bose-Einstein condensates*, SIAM J. Sci. Comput., 36 (2014), B777–B794.
- [49] G. E. KARNIADAKIS, J. S. HESTHAVEN AND I. PODLUBNY, *Fractional PDEs Theory, Numerics and Applications*, J. Comput. Phys., 293 (2015), 1–462.
- [50] K. KIRKPATRICK, E. LENZMANN AND G. STAFFILAN, *On the continuum limit for discrete NLS with long-range lattice interactions*, Commun. Math. Phys., 317 (2012), 563–591.
- [51] K. KIRKPATRICK AND Y. ZHANG, *Fractional Schrödinger dynamics and decoherence*, 2014, preprint.
- [52] C. KLEIN, C. SPARBER AND P. MARKOWICH, *Numerical study of fractional nonlinear Schrödinger equations*, Proc. R. Soc. A, 470 (2014), 20140364.
- [53] N. LASKIN, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, 268 (2000), 298–304.
- [54] N. LASKIN, *Fractional Schrödinger equation*, Phys. Rev. E, 66 (2002), 056108.
- [55] N. LASKIN, *Lévy flights over quantum paths*, Commun. Nonlinear Sci. Numer. Simu., 12 (2007), 2–18.
- [56] N. LASKIN, *Principles of Fractional Quantum Mechanics*, arXiv:1009.5533.
- [57] E. LENZMANN, *Well-posedness for semi-relativistic Hartree equations of critical type*, Math. Phys. Anal. Geom., 10 (2007), 43–64.
- [58] E. LENZMANN, *Uniqueness of ground states for pseudo-relativistic Hartree equations*, Anal. PDE, 1 (2009), 1–30.
- [59] E. H. LIEB AND H. YAU, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys., 112 (1987), 147–174.
- [60] R. METZLER AND J. KLAFTER, *The random walk’s guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., 339 (2000), 1–77.
- [61] R. METZLER AND J. KLAFTER, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A: Math. Gen., 37 (2004), R161–R208.
- [62] J. MING, Q. TANG AND Y. ZHANG, *An efficient spectral method for computing dynamics of rotating two component Bose-Einstein condensates via coordinate transformation*, J. Comput. Phys., 258 (2014), 538–554.
- [63] M. NABER, *Time fractional Schrödinger equation*, J. Math. Phys., 45 (2004), 3339.
- [64] I. PODLUBNY, *Fractional Differential Equations*, Mathematics in Science and Engineering, Volume 198, Academic Press, 1999.
- [65] S. SECCHI, *Ground state solutions for nonlinear fractional Schrödinger equation in \mathbb{R}^N* , J. Math. Phys., 54 (2013), 031501.
- [66] S. SECCHI, *On fractional Schrödinger equation in \mathbb{R}^N without the Ambrosetti-Rabinowitz condition*, arXiv: 1210.0755.
- [67] S. SECCHI AND M. SQUASSINA, *Soliton dynamics for fractional Schrödinger equation*, Appl. Anal., 93 (2014), 1702–1729.
- [68] X. SHANG AND J. ZHANG, *Ground state for fractional Schrödinger equation with critical growth*, Nonlinearity, 27 (2014), 187–207.
- [69] X. SHANG, J. ZHANG AND Y. YANG, *On fractional Schrödinger equation in \mathbb{R}^N with critical growth*, J. Math. Phys., 54 (2013), 121502.
- [70] M. F. SHLESINGER, J. KLAFTER AND G. ZUMOFEN, *Above, below and beyond Brownian motion*, Am. J. Phys., 67 (1999), 1253–1259.
- [71] Q. TANG, *Numerical Studies on Quantized Vortex Dynamics in Superfluidity and Superconductivity*, Ph. D thesis, National University of Singapore, 2013.
- [72] N. UZAR AND S. BALLIKAYA, *Investigation of classical and fractional Bose-Einstein condensation for harmonic potential*, Physica A, 392 (2013), 1733–1741.
- [73] N. UZAR, S. D. HAN, T. TUFEKCI AND E. AYDINER, *Solutions of the Gross-Pitaevskii and time fractional Gross-Pitaevskii equations for different potentials with homotopy perturbation method*, arXiv: 1203.3352.
- [74] L. VLAHOS, H. ISLIKER, Y. KOMINIS AND K. HIZANIDIS, *Normal and Anomalous Diffusion: A Tutorial*, in T. Bountis (ed.) *Order and Chaos*, Vol. 10, Patras University Press, 2008.
- [75] P. WANG AND C. HUANG, *An energy conservative difference scheme for the nonlinear fractional Schrödinger equations*, J. Comput. Phys., 293 (2015), 238–251.
- [76] D. WANG, A. XIAO AND W. YANG, *Crank-Nicolson difference scheme for the coupled nonlinear Schrödinger equations with the Riesz space fractional derivative*, J. Comput. Phys., 242 (2013), 670–681.
- [77] D. WANG, A. XIAO AND W. YANG, *A linearly implicit conservative difference scheme for the space fractional coupled nonlinear Schrödinger equations*, J. Comput. Phys., 272 (2014), 644–655.
- [78] S. WANG AND M. XU, *Generalized fractional Schrödinger equation with space-time fractional derivatives*, J. Math. Phys., 48 (2007), 043502.
- [79] B. J. WEST, M. BOLOGNA AND P. GRIGOLINI, *Physics of Fractal Operators*, Springer, New York, 2003.
- [80] Y. ZHANG AND X. DONG, *On the computation of ground state and dynamics of Schrödinger-Poisson-Slater system*, J. Comput. Phys., 230 (2011), 2660–2676.